RAPID CONSTRUCTION OF REAL NUMBERS BY HALF-CUTS

IWASE, ZJUÑICI

Abstract: This is a rapid construction of the real numbers by using half-cuts of the positive rational numbers, which is based on the idea of Dedekind's cut. Only using these half-cuts, we can naturally define the addition and the multiplication of the positive real numbers and show the completeness of the real numbers.

1. Half Cuts of $\mathbb{Q}_+$

Let $\mathbb{Q}_+$ be the set of all the positive rational numbers.

Definition 1.1 A subset A of $\mathbb{Q}_+$ is called a *half cut* of $\mathbb{Q}_+$ (or just a *half cut*) if the following conditions are satisfied. (i) A is not empty. (ii) A has an *upper bound* $M \in \mathbb{Q}_+$, *i.e.* if $a \in A$ then $a \leq M$. (iii) If $a \in A$ and $a' \in \mathbb{Q}_+$ with $a' < a$, then $a' \in A$. (iv) A does not have a maximum number.

The set of all the half cuts of $\mathbb{Q}_+$ is denoted by X_+.

Definition 1.2 For two half cuts A and B, define $A \leq B$ if $A \subset B$. Define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 1.3 The relation $\leq$ is a total order on X_+, *i.e.* for half cuts A, B and C, we have (i) $A \leq A$, (ii) if $A \leq B$ and $B \leq A$ then $A = B$, (iii) if $A \leq B$ and $B \leq C$ then $A \leq C$, (iv) $A \leq B$ or $B \leq A$.

Proof. (i), (ii), (iii) are clear. (iv) Assume that $A \not\leq B$. There is a number r with $r \in A$ and $r \notin B$. Since A is a half cut, therefore if $s \in \mathbb{Q}_+$ and $s < r$ then $s \in A$. Since B is a half cut, therefore if $s \in B$ then $s < r$. Hence $B \leq A$ holds. □

Definition 1.4 Define the map $\iota : \mathbb{Q}_+ \to X_+$ by $\iota(r) = \{a \in \mathbb{Q}_+ | a < r\}$.

Fact 1.5 The map ι preserves order, *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Corollary 1.6 The map ι is injective.

2. Addition and Multiplication of Half Cuts of $\mathbb{Q}_+$

Definition 2.1 For half cuts A and B, put $A + B = \{a + b | a \in A,\ b \in B\}$ and put $AB = \{ab | a \in A,\ b \in B\}$.

Lemma 2.2 If A and B are half cuts, then $C = A + B$ (*resp.* $C = AB$) is a half cut.

Proof. (i) Since A and B are non-empty, therefore C is not empty. (ii) Since A and B have upper bounds, therefore C has an upper bound. (iii) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). If $c' \in \mathbb{Q}_+$ and $c' \leq c$, then $ac'/c \in A$ and $bc'/c \in B$. Since $c' = ac'/c + bc'/c$ (*resp.* $c' = (ac'/c)b$), therefore $c' \in C$. (iv) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). There exists $a' \in A$ with $a < a'$. Then, $c' = a' + b$ (*resp.* $c' = a'b$) is an element of C with $c < c'$. □

Fact 2.3 Let A, B and C be half cuts. Then $A+B = B+A$, $(A+B)+C = A+(B+C)$, $AB = BA$, $(AB)C = A(BC)$, $A(B+C) = AB+AC$, $(A+B)C = AC+BC$.

Lemma 2.4 The addition and multiplication defined above commute with the inclusion map ι, *i.e.* if r, $s \in \mathbb{Q}_+$, then $\iota(r + s) = \iota(r) + \iota(s)$ and $\iota(rs) = \iota(r)\iota(s)$.

Proof. $\iota(r+s) \supset \iota(r)+\iota(s)$ (*resp.* $\iota(rs) \supset \iota(r)\iota(s)$) is clear. Let x be an element of $\iota(r+s)$ (*resp.* $\iota(rs)$). Since $xr/(r+s) < r$ and $xs/(r+s) < s$ (*resp.* There exists $q \in \mathbb{Q}_+$ with $x/(sr) < q < 1$. Since $qr < r$ and $x/(qr) < s$), therefore $\iota(r+s) \subset \iota(r)+\iota(s)$ (*resp.* $\iota(rs) \subset \iota(r)\iota(s)$).

Lemma 2.5 Let A, B and C be half cuts of $\mathbb{Q}_+$. If $A < B$, then $A+C < B+C$ (*resp.* $AC < BC$).

Proof. Let r be a number with $r \notin A$ and $r \in B$. Let r' be a number with $r < r'$ and $r' \in B$. There exists a number s with $s \in C$ and $s+(r'-r) \notin C$ (*resp.* $s(r'/r) \notin C$). Put $s' = s+(r'-r)$ (*resp.* $s' = s(r'/r)$). The number $r'+s$ (*resp.* $r's$) is an element of $B+C$ (*resp.* BC). Put $t = r'+s$ (*resp.* $t = r's$), which is equal to $r+s'$ (*resp.* rs'). Since $r \notin A$ and $s' \notin C$, therefore $t \notin A+C$ (*resp.* $t \notin AC$), because if there exist $r'' \in A$ and $s'' \in C$ with $t = r''+s''$ (*resp.* $t = r''s''$), then $r < r''$ or $s' < s''$ must hold. This is a contradiction. □

Lemma 2.6 For any half cut A, we have $A\iota(1) = \iota(1)A = A$.

Proof. $A\iota(1) \subset A$ is clear. Assume that $a \in A$. There exists an element $a' \in A$ with $a < a'$. Since $a/a' < 1$, therefore $a/a' \in \iota(1)$. Hence $a = a'(a/a') \in A\iota(1)$. □

Lemma 2.7 The multiplication on X_+ is Archimedean, *i.e.* for any two half cuts A and B, there exists a natural number n such that $A < \iota(n)B$.

Proof. Let M be an upper bound of A and let b be an element of B. Since $\mathbb{Q}_+$ is Archimedean, there exists a natural number n with $M < nb$. □

3. Completeness of Positive Real Numbers

A member of X_+ is called a positive *real number*. Denote X_+ by $\mathbb{R}_+$.

Let Λ be a non-empty subset of $\mathbb{R}_+$. A positive real number α is called a *supremum* of Λ if it satisfies the following conditions. (i) If $A \in \Lambda$, then $A \leq \alpha$. (ii) If $\alpha' < \alpha$ and $\alpha' \in \mathbb{R}_+$, then there exists $A \in \Lambda$ with $\alpha' < A$.

Let Λ be a non-empty subset of $\mathbb{R}_+$ and M be an element of $\mathbb{R}_+$. We call M an *upper bound* of Λ if $A \in \Lambda$ implies $A \leq M$.

Propositon 3.1 Let Λ be a non-empty subsets of $\mathbb{R}_+$ with an upper bound. Then Λ has a supremum.

Proof. Let α be the set $\cup_{A\in\Lambda} A$. Then, α is a half cut of $\mathbb{Q}_+$. In fact, (i) α is not empty. (ii) α has an upper bound. (iii) If $a \in \alpha$ and $a' \in \mathbb{Q}_+$ with $a' < a$, then $a' \in \alpha$ holds. In fact, there exists $A \in \Lambda$ with $a \in A$. Then, $a' \in A$. Hence $a' \in \alpha$. (iv) α does not have a maximum number. In fact, if $a \in \alpha$, then there exists $A \in \Lambda$ with $a \in A$. Since A is a half cut, therefore there exists $a' \in A \subset \alpha$ with $a < a'$.

We shall prove that α is a supremum of Λ. If $A \in \Lambda$, then $A \leq \alpha$. If $\alpha' < \alpha$, then there exists $r \in \alpha$ with $r \notin \alpha'$. There exists $A \in \Lambda$ with $r \in A$. Since $\alpha' < A$, we have the conclusion. □

4. Constructing Real Numbers from $\mathbb{R}_+$

Constructing real numbers from $\mathbb{R}_+$ is same as the construction of $\mathbb{Z}$ from $\mathbb{N}$.

Departments of the School of Mathematics and Physics, College of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa, 920-1192, Japan.

E-mail address: iwase@staff.kanazawa-u.ac.jp