RAPID CONSTRUCTION OF REAL NUMBERS BY HALF-CUTS

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Abstract: A rapid construction of the real numbers by using half-cuts of the positive rational numbers which is based on the idea of Dedekind's cut. We can naturally define the addition and the multiplication of the positive real numbers and show the completeness of the real numbers.

1. Half-Cuts of $\mathbb{Q}_+$

Let $\mathbb{Q}_+$ be the set of all the positive rational numbers.

Fact 1.1 If a, $b \in \mathbb{Q}_+$, then $(a+b)/2 \in \mathbb{Q}_+$. Therefore, if c, $d \in \mathbb{Q}_+$ with $c < d$, then there is an elemant $x \in \mathbb{Q}_+$ such that $c < x < d$.

Definition 1.2 A subset A of $\mathbb{Q}_+$ is called a *half-cut* of $\mathbb{Q}_+$ (or just a *half-cut*) if the following conditions are satisfied. (i) A is not empty. (ii) A has an *upper bound* $M \in \mathbb{Q}_+$, *i.e.* if $a \in A$ then $a \leq M$. (iii) If $a \in A$ and $a' \in \mathbb{Q}_+$ with $a' < a$, then $a' \in A$. (iv) A does not have a maximum number.

The set of all the half-cuts of $\mathbb{Q}_+$ is denoted by X_+.

Definition 1.3 For two half-cuts A and B, define $A \leq B$ if $A \subset B$. Define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 1.4 The relation $\leq$ is a total order on X_+, *i.e.* for half-cuts A, B and C, we have (i) $A \leq A$, (ii) if $A \leq B$ and $B \leq A$ then $A = B$, (iii) if $A \leq B$ and $B \leq C$ then $A \leq C$, (iv) $A \leq B$ or $B \leq A$.

Proof. (i), (ii), (iii) are clear. (iv) Assume that $A \not\leq B$. There is a number r with $r \in A$ and $r \notin B$. Since A is a half-cut, therefore if $s \in \mathbb{Q}_+$ and $s < r$ then $s \in A$. Since B is a half-cut, therefore if $s \in B$ then $s < r$. Hence $B \leq A$ holds. □

Definition 1.5 Define the map $\iota : \mathbb{Q}_+ \to X_+$ by $\iota(r) = \{a \in \mathbb{Q}_+ | a < r\}$.

Fact 1.6 The map ι preserves order, *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Lemma 1.7 Assume that r, s and $c \in \mathbb{Q}_+$ with $c < r + s$ (*resp.* $c < rs$). Then, there exist r', $s' \in \mathbb{Q}_+$ with $r' < r$, $s' < s$, and $c = r' + s'$ (*resp.* $c = r's'$).

Proof. We must find $r' \in \mathbb{Q}_+$ with $c - s < r' < r$ (*resp.* $c/s < r' < r$). Apply Fact 1.1. □

2. Addition and Multiplication of Half-Cuts of $\mathbb{Q}_+$

Definition 2.1 For half-cuts A and B, put $A + B = \{a + b | a \in A,\ b \in B\}$ and put $AB = \{ab | a \in A,\ b \in B\}$.

Lemma 2.2 If A and B are half-cuts, then $C = A + B$ (*resp.* $C = AB$) is a half-cut.

Proof. (i) Since A and B are non-empty, therefore C is not empty. (ii) Since A and B have upper bounds, therefore C has an upper bound. (iii) See Lemma 1.7. (iv) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). There exists $a' \in A$ with $a < a'$. Then, $c' = a' + b$ (*resp.* $c' = a'b$) is an element of C with $c < c'$. □

Fact 2.3 Let A, B and C be half-cuts. Then $A+B=B+A$, $(A+B)+C=A+(B+C)$, $AB=BA$, $(AB)C=A(BC)$, $A(B+C)=AB+AC$, $(A+B)C=AC+BC$.

Proof. We shall prove $A(B+C)\supset AB+AC$. If $x\in AB+AC$, then there exist a, $a'\in A$, $b\in B$, and $c\in C$ such that $x=ab+a'c$. If $a<a'$, then $x'=a'b+a'c=a'(b+c)\in A(B+C)$ and $x<x'$. □

Lemma 2.4 Let A, B and C be half-cuts of $\mathbb{Q}_+$. If $A<B$, then $A+C<B+C$ (*resp.* $AC<BC$).

Proof. Assume that r, $r+\delta\in B\setminus A$ and $\delta>0$ (*resp.* r, $r\rho\in B\setminus A$ and $\rho>1$). There exists an element $s\in C$ with $s+\delta\notin C$ (*resp.* $s\rho\notin C$). Note that $(r+\delta)+s\in B+C$ (*resp.* $(r\rho)s\in BC$) and $r+(\delta+s)\notin A+C$ (*resp.* $r(\rho s)\notin AC$). Because if there exist $r'\in A$ and $s'\in C$ with $r+\delta+s=r'+s'$ (*resp.* $(r\rho s=r's')$, then $r<r'$ or $s<s'$ must hold. This is a contradiction. □

Lemma 2.5 The addition and multiplication defined above commute with the inclusion map ι, *i.e.* if r, $s\in\mathbb{Q}_+$, then $\iota(r+s)=\iota(r)+\iota(s)$ and $\iota(rs)=\iota(r)\iota(s)$.

Proof. $\iota(r+s)\supset\iota(r)+\iota(s)$ (*resp.* $\iota(rs)\supset\iota(r)\iota(s)$) is clear. Assume that $x\in\iota(r+s)$, *i.e.* $x<r+s$ (*resp.* $x\in\iota(rs)$, *i.e.* $x<rs$). Apply Lemma 1.7. □

Lemma 2.6 For any half-cut A, we have $A\iota(1)=\iota(1)A=A$.

Proof. $A\iota(1)\subset A$ is clear. Assume that $a\in A$. There exists an element $a'\in A$ with $a<a'$. We have $a<a'1$. Apply Lemma1.7. □

Lemma 2.7 The multiplication on X_+ is Archimedean, *i.e.* for any two half-cuts A and B, there exists a natural number n such that $A<\iota(n)B$.

Proof. Let M be an upper bound of A and let b be an element of B. Since $\mathbb{Q}_+$ is Archimedean, there exists a natural number n with $M<nb$. □

3. Completeness of Positive Real Numbers

A member of X_+ is called a positive *real number*. Denote X_+ by $\mathbb{R}_+$.

Let Λ be a non-empty subset of $\mathbb{R}_+$. A positive real number α is called a *supremum* of Λ if it satisfies the following conditions. (i) If $A\in\Lambda$, then $A\le\alpha$. (ii) If $\alpha'<\alpha$ and $\alpha'\in\mathbb{R}_+$, then there exists $A\in\Lambda$ with $\alpha'<A$.

Let Λ be a non-empty subset of $\mathbb{R}_+$ and M be an element of $\mathbb{R}_+$. We call M an *upper bound* of Λ if $A\in\Lambda$ implies $A\le M$.

Propositon 3.1 Let Λ be a non-empty subsets of $\mathbb{R}_+$ with an upper bound. Then Λ has a supremum.

Proof. Let α be the set $\cup_{A\in\Lambda}A$. Then, α is a half-cut of $\mathbb{Q}_+$. In fact, (i) α is not empty. (ii) α has an upper bound. (iii) If $a\in\alpha$ and $a'\in\mathbb{Q}_+$ with $a'<a$, then $a'\in\alpha$ holds. In fact, there exists $A\in\Lambda$ with $a\in A$. Then, $a'\in A$. Hence $a'\in\alpha$. (iv) α does not have a maximum number. In fact, if $a\in\alpha$, then there exists $A\in\Lambda$ with $a\in A$. Since A is a half-cut, therefore there exists $a'\in A\subset\alpha$ with $a<a'$.

We shall prove that α is a supremum of Λ. If $A\in\Lambda$, then $A\le\alpha$. If $\alpha'<\alpha$, then there exists $r\in\alpha$ with $r\notin\alpha'$. There exists $A\in\Lambda$ with $r\in A$. Since $\alpha'<A$, we have the conclusion. □

4. Constructing Real Numbers from $\mathbb{R}_+$

Constructing real numbers from $\mathbb{R}_+$ is same as the construction of $\mathbb{Q}$ from $\mathbb{N}$.

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