QUICK CONSTRUCTION OF REAL NUMBERS BY HALF-CUTS

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Abstract: A quick construction of the real numbers by using half-cuts of the positive finite decimals which is based on the idea of Dedekind's cut. We can naturally define the addition and the multiplication of the positive real numbers and show the completeness of the real numbers.

1. Half-Cuts of the Set of all the Positive Finite Decimals

Let F_+ be the set of all the positive finite decimals. Put $F_+^\times = \{x \in F_+ | x^{-1} \in F_+\}$. Note that $F_+^\times = \{2^m 5^n | m, n \in \mathbb{Z}\}$.

Fact 1.1 If a, $b \in F_+$, then $(a+b)/2 \in F_+$. Therefore, if c, $d \in F_+$ with $c < d$, then there is an element $x \in F_+$ such that $c < x < d$.

Definition 1.2 A subset A of F_+ is called a *half-cut* of F_+ (or just a *half-cut*) if the following conditions are satisfied. (i) A is not empty. (ii) A has an *upper bound* $M \in F_+$, *i.e.* if $a \in A$ then $a \leq M$. (iii) If $a \in A$ and $a' \in F_+$ with $a' < a$, then $a' \in A$. (iv) A does not have a maximum number.

The set of all the half-cuts of F_+ is denoted by X_+.

Definition 1.3 For two half-cuts A and B, define $A \leq B$ if $A \subset B$. Define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 1.4 The relation $\leq$ is a total order on X_+, *i.e.* for half-cuts A, B and C, we have (i) $A \leq A$, (ii) if $A \leq B$ and $B \leq A$ then $A = B$, (iii) if $A \leq B$ and $B \leq C$ then $A \leq C$, (iv) $A \leq B$ or $B \leq A$.

Proof. (i), (ii), (iii) are clear. (iv) Assume that $A \not\leq B$. There is a number r with $r \in A$ and $r \notin B$. Since A is a half-cut, therefore if $s \in F_+$ and $s < r$ then $s \in A$. Since B is a half-cut, therefore if $s \in B$ then $s < r$. Hence $B \leq A$ holds. □

Definition 1.5 Define the map $\iota : F_+ \to X_+$ by $\iota(r) = \{a \in F_+ | a < r\}$.

Fact 1.6 The map ι preserves order, *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Lemma 1.7 Assume that r, s and $c \in F_+$ with $c < r + s$ (*resp.* $c < rs$). Then, there exist r', $s' \in F_+$ with $r' < r$, $s' < s$, and $c = r' + s'$ (*resp.* $c = r's'$).

Proof. We must find $r' \in F_+$ with $c - s < r' < r$. Apply Fact 1.1 (*resp.* We must find $r' \in F_+^\times$ with $c/s < r' < r$. See the sublemmata below). □

Sublemma 1.8 If $\rho > 1$, then, these exist integers m, n such that $1 < 2^m 5^n < \rho$.

Proof. Consider the set $\{2^m 5^n | 1 \leq 2^m 5^n < 5, m = 1, 2, 3, ..., N, n \in \mathbb{Z}\}$. This set consists of N elements, by the uniqueness of the prime factorization.

Put its members in the increasing order, consider the ratio of all the pairs of the adjacent members of this set. Then, there exists a pair $(2^m 5^n,\ 2^{m'} 5^{n'})$ such that their ratio r satisfies $1 < r^{N-1} < 5$. If N is sufficiently large, then we have $1 < r < \rho$, which is equal to $1 < (2^m 5^n)/(2^{m'} 5^{n'}) < \rho$, and $1 < 2^{m-m'} 5^{n-n'} < \rho$. □

Sublemma 1.9 Any interval I of F_+ contains an element of $F_+^\times$.

Proof. Let x, y be the members of I with $x < y$. Put $\rho = y/x$, apply Sublemma 1.8. Then we obtain a member r of $F_+^\times$. The set $\{r^i | i \in \mathbb{Z}\} \cap I$ is not empty.

Remark 1.10 $F_+^\times$ is a dense subset of $\mathbb{Q}_+$. It is a subgroup of $(\mathbb{Q}_+, \times)$.

Example 1.11 Assume that $r = 0.2$, $s = 0.3$ and $c = 0.05$. Then, $r' = 0.1953125$, and $s' = 0.256$, *i.e.* $c = 0.05 = 0.1953125 \times 0.256 = r's'$.

2. Addition and Multiplication of Half-Cuts of F_+

Definition 2.1 For half-cuts A and B, put $A + B = \{a + b | a \in A,\ b \in B\}$ and put $AB = \{ab | a \in A,\ b \in B\}$.

Lemma 2.2 If A and B are half-cuts, then $C = A + B$ (*resp.* $C = AB$) is a half-cut.

Proof. (i) Since A and B are non-empty, therefore C is not empty. (ii) Since A and B have upper bounds, therefore C has an upper bound. (iii) Apply Lemma 1.7. (iv) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). There exists $a' \in A$ with $a < a'$. Then, $c' = a' + b$ (*resp.* $c' = a'b$) is an element of C with $c < c'$. □

Fact 2.3 Let A, B and C be half-cuts. Then $A+B = B+A$, $(A+B)+C = A+(B+C)$, $AB = BA$, $(AB)C = A(BC)$, $A(B+C) = AB+AC$, $(A+B)C = AC+BC$.

Proof. We shall prove $A(B + C) \supset AB + AC$. If $x \in AB + AC$, then there exist a, $a' \in A$, $b \in B$, and $c \in C$ such that $x = ab + a'c$. If $a < a'$, then $x' = a'b + a'c = a'(b + c) \in A(B + C)$ and $x < x'$. □

Lemma 2.4 Let A, B and C be half-cuts of F_+. If $A < B$, then $A+C < B+C$ (*resp.* $AC < BC$).

Proof. Assume that r, $r + \delta \in B \setminus A$ and $\delta > 0$ (*resp.* r, $r\rho \in B \setminus A$ and $\rho > 1$). There exists an element $s \in C$ with $s + \delta \notin C$ (*resp.* $s\rho \notin C$). Note that $(r+\delta)+s \in B+C$ (*resp.* $(r\rho)s \in BC$) and $r+(\delta+s) \notin A+C$ (*resp.* $r(\rho s) \notin AC$). Because if there exist $r' \in A$ and $s' \in C$ with $r + \delta + s = r' + s'$ (*resp.* $r\rho s = r's'$), then $r < r'$ or $s < s'$ must hold. This is a contradiction. □

Lemma 2.5 The addition and multiplication defined above commute with the inclusion map ι, *i.e.* if r, $s \in F_+$, then $\iota(r + s) = \iota(r) + \iota(s)$ and $\iota(rs) = \iota(r)\iota(s)$.

Proof. $\iota(r + s) \supset \iota(r) + \iota(s)$ (*resp.* $\iota(rs) \supset \iota(r)\iota(s)$) is clear. Assume that $x \in \iota(r + s)$, *i.e.* $x < r + s$ (*resp.* $x \in \iota(rs)$, *i.e.* $x < rs$). Apply Lemma 1.7. □

Lemma 2.6 For any half-cut A, we have $A\iota(1) = \iota(1)A = A$.

Proof. $A\iota(1) \subset A$ is clear. Assume that $a \in A$. There exists an element $a' \in A$ with $a < a'$. We have $a < a'1$. Apply Lemma1.7. □

Lemma 2.7 The multiplication on X_+ is Archimedean, *i.e.* for any two half-cuts A and B, there exists a natural number n such that $A < \iota(n)B$.

Proof. Let M be an upper bound of A and let b be an element of B. Since $\mathbb{Q}_+$ is Archimedean, there exists a natural number n with $M < nb$. □

3. Completeness of Positive Real Numbers

See [1].

4. Constructing Real Numbers from $\mathbb{R}_+$

See [1].

References

[1] Rapid construction of real numbers by half-cuts (preprint).

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