CONSTRUCTING REAL NUMBERS BY DEDEKIND'S CUTS IN A SLIGHTLY MODIFIED WAY

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Let $\mathbb{Q}$ be the set of all the rational numbers. Let $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) be the set of all the positive (resp. negative) rational numbers.

1. Cuts of $\mathbb{Q}$

Definition 1.1 The ordered pair $\alpha = (A_0, A_1)$ of subsets of $\mathbb{Q}$ is called a *cut of* $\mathbb{Q}$ (or just a *cut*) if the following conditions are satisfied.

(i) $A_0 \neq \emptyset$ and $A_1 \neq \emptyset$.

(ii) $A_0 \cup A_1 = \mathbb{Q}$ or $\mathbb{Q} \setminus \{r\}$, where r is a rational number.

(iii) If $a_0 \in A_0$ and $a_1 \in A_1$, then $a_0 < a_1$ holds. Moreover, if $A_0 \cup A_1 = \mathbb{Q} \setminus \{r\}$, then $a_0 < r < a_1$ holds.

(iv) A_0 does not have a maximum number. A_1 does not have a minimum number.

The set of all the cuts of $\mathbb{Q}$ is denoted by X.

Let S be a subset of $\mathbb{Q}$. Put $-S = \{-s | s \in S\}$.

Lemma 1.2 Let $\alpha = (A_0, A_1)$ be any cut. Then, $(-A_1, -A_0)$ is a cut. If $A_0 \cup A_1 = \mathbb{Q} \setminus \{r\}$, then $(-A_1) \cup (-A_0) = \mathbb{Q} \setminus \{-r\}$.

Proof. (i) There exists an element a_0 in A_0. Since $-a_0 \in -A_0$, therefore, we have $-A_0 \neq \emptyset$. Similarly, we have $-A_1 \neq \emptyset$.

(ii) Assume that $A_0 \cup A_1 = \mathbb{Q}$. Let s be any rational number. Then, $-s \in A_0$ or $-s \in A_1$ holds. This means that $s \in -A_0$ or $s \in -A_1$. Hence $(-A_1) \cup (-A_0) = \mathbb{Q}$.

Assume that $A_0 \cup A_1 = \mathbb{Q} \setminus \{r\}$. Let s be any rational number with $s \neq -r$. Then, $-s \in A_0$ or $-s \in A_1$ holds. This means that $s \in -A_0$ or $s \in -A_1$. If $-r \in (-A_1) \cup (-A_0)$, then we have $r \in A_0 \cup A_1$. This is a contradiction. Hence $(-A_1) \cup (-A_0) = \mathbb{Q} \setminus \{-r\}$.

(iii) Assume that $a_0 \in -A_0$ and that $a_1 \in -A_1$. Since $-a_0 \in A_0$ and $-a_1 \in A_1$, therefore $-a_0 < -a_1$ holds. Hence we have $a_1 < a_0$. Moreover, assume that $A_0 \cup A_1 = \mathbb{Q} \setminus \{r\}$. Since $-a_0 < r < -a_1$, therefore we have $a_1 < -r < a_0$.

(iv) Let a_0 be an element of $-A_1$. Then, $-a_0$ is an element of A_1. Since A_1 does not have a minimum number, therefore there exists an element a' in A_1 with $a' < -a_0$. Since $-a'$ is an element of $-A_1$ and $-a' > a_0$, therefore a_0 is not a maximum number of $-A_1$. We know that $-A_1$ does not have a maximum number. Similarly, we can prove that $-A_0$ does not have a minimum number. □

Remark Lemma 1.2 tells that the definition of a cut in Definition 1.1 is "symmetric".

Definition 1.3 For any cut $\alpha = (A_0, A_1)$, put $-\alpha = (-A_1, -A_0)$.

Remark 1.4 Assume that $\alpha = (A_0, A_1)$ is a cut.

(i-a) $A_1 = \mathbb{Q} \setminus A_0$ if $\mathbb{Q} \setminus A_0$ does not have a minimum number. If $\mathbb{Q} \setminus A_0$ has a minimum number r, then $A_1 = \mathbb{Q} \setminus (A_0 \cup \{r\})$. This means that A_1 is determined by A_0.

(i-b) $A_0 = \mathbb{Q} \setminus A_1$ if $\mathbb{Q} \setminus A_1$ does not have a maximum number. If $\mathbb{Q} \setminus A_1$ has a maximum number r, then $A_0 = \mathbb{Q} \setminus (A_1 \cup \{r\})$. This means that A_0 is determined by A_1.

(ii-a) If $a \in A_0$ and $a' \leq a$, then $a' \in A_0$ holds.

(ii-a') If $a' \notin A_0$ and $a' \leq a$, then $a \notin A_0$ holds.

(ii-a") If $a \in A_0$ and $a' \notin A_0$, then $a < a'$ holds.

(ii-b) If $a \in A_1$ and $a \leq a'$, then $a' \in A_1$ holds.

(ii-b') If $a' \notin A_1$ and $a \leq a'$, then $a \notin A_1$ holds.

(ii-b") If $a' \in A_1$ and $a \notin A_1$, then $a < a'$ holds.

Lemma 1.5 For two cuts $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$, $A_0 \subsetneq B_0$ if and only if $A_1 \supsetneq B_1$.

Proof. Assume that $A_0 \subsetneq B_0$. Then, there exists a rational number r such that $r \notin A_0$ and $r \in B_0$. Let x be any rational number in B_1. Since (B_0, B_1) is a cut and $r \in B_0$ and $x \in B_1$, therefore we have $r < x$. By Remark 1.4 (ii-a') and the fact that $r \notin A_0$ and that $r < x$, we have $x \notin A_0$. If $x \notin A_1$, then, by Remark 1.4 (ii-b'), we have $r \notin A_1$. Hence r, $x \notin A_0 \cup A_1$ holds. This contradicts to the definition of the cut. Therefore, $x \in A_1$ holds. We have proved $B_1 \subset A_1$.

If $B_1 = A_1$, then, by Remark 1.4 (i-b), $A_0 = B_0$ holds. This is a contradiction. We have proved that $B_1 \subsetneq A_1$.

The converse is proved similarly. □

Lemma 1.6 Let $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ be two cuts. If $A_0 \subset B_0$ and $A_1 \subset B_1$, then $\alpha = \beta$ holds.

Proof. If $A_0 \subsetneq B_0$ and $A_1 \subsetneq B_1$, then, there exist two rational numbers r_0 and r_1 with $r_0 \in B_0 \setminus A_0$, $r_1 \in B_1 \setminus A_1$. Since $r_0 \in B_0$ and $r_1 \in B_1$, therefore $r_0 < r_1$. Since $r_0 \notin A_0$ and $r_1 \notin A_1$ and $r_0 < r_1$, therefore $r_1 \notin A_0$ by Remark 1.4 (ii-a') and $r_0 \notin A_1$ by Remark 1.4 (ii-b'). Hense $\mathbb{Q} \setminus (A_0 \cup A_1)$ contains two distinct rational numbers r_0, r_1. This is a contradiction. Therefore, we have $A_0 = B_0$ or $A_1 = B_1$. By Remark 1.4 (i-a) or (i-b), we have the conclusion. □

2. Order on the set of all the cuts of $\mathbb{Q}$

Definition 2.1 For two cuts $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$, define $\alpha \leq \beta$ if $A_0 \subset B_0$. Define $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

We also use the notation $\beta \geq \alpha$ (resp. $\beta > \alpha$), which is equivalent to $\alpha \leq \beta$ (resp. $\alpha < \beta$).

Proposition 2.2 The relation $\leq$ is a total order on X. *i.e.*

(1) For any cut α, we have $\alpha \leq \alpha$.

(2) For any cuts α and β, if $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$ holds.

(3) For any cuts α, β and γ, if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ holds.

(4) For any cuts α and β, we have $\alpha \leq \beta$ or $\beta \leq \alpha$.

Proof. Put $\alpha = (A_0, A_1)$, $\beta = (B_0, B_1)$, $\gamma = (C_0, C_1)$.

(1) Since $A_0 \subset A_0$, therefore we have $\alpha \leq \alpha$.

(2) Since $A_0 \subset B_0$ and $B_0 \subset A_0$, therefore we have $A_0 = B_0$. Hence $\alpha = \beta$ by Remark 1.4 (i-a).

(3) Since $A_0 \subset B_0$ and $B_0 \subset C_0$, therefore we have $A_0 \subset C_0$. By definition, $\alpha \leq \gamma$ holds.

(4) Assume that $\alpha \not\leq \beta$. Since $A_0 \not\subset B_0$, therefore there is a rational number r with $r \in A_0$ and $r \notin B_0$. Let x be any element in B_0. By Remark 1.4 (ii-a"), we

have $x < r$. By Remark 1.4 (ii-a) and the fact $r \in A_0$, we have $x \in A_0$. We have proved $B_0 \subset A_0$. Hence $\beta \leq \alpha$ holds. □

3. Inclusion map from $\mathbb{Q}$ to the set of all the cuts of $\mathbb{Q}$

Definition 3.1 Define the map $\iota : \mathbb{Q} \to X$ by $\iota(r) = (] -\infty, r[,]r, +\infty[)$.

Lemma 3.2 The map ι preserves order. *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Proof. Assume that $r < s$. Denote $\iota(r)$ by (R_0, R_1) and $\iota(s)$ by (S_0, S_1). Note that $r \in S_0$ and $r \notin R_0$. By Proposition 2.2 (4), we have $\iota(r) < \iota(s)$. □

Corollary 3.3 The map ι is injective.

4. Addition of cuts of $\mathbb{Q}$

Let S and T be subsets of rational numbers. Put $S + T = \{s + t | s \in S, t \in T\}$.

Proposition 4.1 For two cuts $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$, put $C_0 = A_0 + B_0$, $C_1 = A_1 + B_1$. Then, the following holds.

(a) (i) If $x \in C_0$ and a rational number y satisfies $y \leq x$, then we have $y \in C_0$. (ii) If $x \in C_1$ and a rational number y satisfies $y \geq x$, then we have $y \in C_1$.

(b) (C_0, C_1) is a cut.

Proof. (a) (i) Since $x \in C_0$, therefore there exist rational numbers a and b such that $a \in A_0$ and $b \in B_0$ with $c = a + b$. Put $b' = b - (x - y)$. Since $b' \leq b$ and $b \in B_0$, therefore we have $b' \in B_0$ by Remark 1.4 (ii-a). Since $y = a + b'$, therefore we have $y \in C_0$. (ii) can be proved similarly.

(b) (i) Since A_0 and B_0 are non-empty set, therefore there exist rational numbers a_0 and b_0 such that $a_0 \in A_0$ and $b_0 \in B_0$. Since $a_0 + b_0$ is an element of $A_0 + B_0 = C_0$, therefore we have that C_0 is not an empty set. Similarly, we can show that C_1 is not empty.

(ii) Assume that $\mathbb{Q} \setminus (C_0 \cup C_1)$ contains more than one rational number, x, y with $x < y$. Assume that z is a rational number with $x < z < y$. If $z \in C_0$, then $x < z$ contradicts to (a)(i). If $z \in C_1$, then $z < y$ contradicts to (a)(ii). We have proved that $x < z < y$ implies $z \notin C_0 \cup C_1$.

Put $r = (y - x)/4$. It is a positive rational number. Consider the set of rational numbers $S = \{\cdots, -4r, -3r, -2r, -r, 0, r, 2r, 3r, 4r, \cdots\}$. There exists an integer m such that $\{\cdots, (m-3)r, (m-2)r, (m-1)r\} \subset A_0$ and $\{(m+1)r, (m+2)r, (m+3)r, \cdots\} \subset A_1$. Similarly, there exists an integer n such that $\{\cdots, (n-3)r, (n-2)r, (n-1)r\} \subset B_0$ and $\{(n+1)r, (n+2)r, (n+3)r, \cdots\} \subset B_1$. Then, $\{\cdots, (m+n-4)r, (m+n-3)r, (m+n-2)r\} \subset C_0$ and $\{(m+n+2)r, (m+n+3)r, (m+n+4)r, \cdots\} \subset C_1$. Note that at most three elements of S, namely, $(m+n-1)r$, $(m+n)r$, $(m+n+1)r$, are not contained $C_0 \cup C_1$. Since $y - x = 4r$, this is a contradiction. We have shown that $\mathbb{Q} \setminus (C_0 \cup C_1)$ consists of at most one rational number.

(iii) For $i = 0$, 1, let c_i be an element of C_i. Then there exists an element $a_i \in A_i$ and an element $b_i \in B_i$ such that $a_i + b_i = c_i$. Since $a_0 < a_1$ and $b_0 < b_1$, therefore $c_0 < c_1$ holds.

Assume that $\mathbb{Q} \setminus (C_0 \cup C_1) = \{r\}$. Assume that $c_0 \in C_0$ and $c_0 \geq r$. Then we have $r \in C_0$ by (a)(i). This is a contradiction. Therefore we have $c_0 < r$. Assume that $c_1 \in C_1$ and $r \geq c_1$. Then we have $r \in C_1$ by (a)(ii). This is a contradiction. Therefore we have $r < c_1$.

(iv) Let c_0 be an element of C_0. There exists an element $a_i \in A_i$ and an element $b_i \in B_i$ such that $a_i + b_i = c_i$. Since a_0 is not a maximum number of A_0, therefore there exists an element a_0' of A_0 with $a_0 < a_0'$. Since b_0 is not a maximum number of B_0, therefore there exists an element b'_0 of B_0 with $b_0 < b'_0$. Since $a'_0 + b'_0$ is an element of C_0 and greater than $a_0 + b_0 = c_0$, therefore c_0 cannot be a maximum number of C_0. Hence C_0 does not have a maximum number. Similarly, we can prove that C_1 does not have a minimum number. □

Definition 4.2 For two cuts $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$, define $\alpha + \beta = (A_0 + B_0, A_1 + B_1)$.

By Proposition 4.1, this is a binary operation on X.

Lemma 4.3 Let $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ be two cuts of $\mathbb{Q}$. Then $\alpha + \beta = \beta + \alpha$ holds.

Proof. Since $A_i + B_i = B_i + A_i$ for $i = 0, 1$, therefore we have the conclusion. □

Lemma 4.4 Let $\alpha = (A_0, A_1)$, $\beta = (B_0, B_1)$ and $\gamma = (C_0, C_1)$ be three cuts of $\mathbb{Q}$. Then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ holds.

Proof. Since $(A_i + B_i) + C_i = A_i + (B_i + C_i)$ for $i = 0, 1$, therefore we have the conclusion. □

Lemma 4.5 The addition defined above commutes with the inclusion map ι, *i.e.* for any rational numbers r, s, we have $\iota(r + s) = \iota(r) + \iota(s)$.

Proof. Put $\iota(r) = (R_0, R_1)$, $\iota(s) = (S_0, S_1)$, $\iota(r + s) = (T_0, T_1)$. If $x \in R_0$ and $y \in S_0$, then $x < r$ and $y < s$ hold. Since $x + y < r + s$, therefore $x + y \in T_0$. This means that $R_0 + S_0 \subset T_0$. Similarly, we can show that $R_1 + S_1 \subset T_1$. By Lemma 1.6, we have the conclusion. □

Lemma 4.6 $\alpha + \iota(0) = \iota(0) + \alpha = \alpha$ holds.

Proof. Put $O_0 =]-\infty, 0[$ and $O_1 =]0, +\infty[$. Note that $\iota(0) = (O_0, O_1)$. By Remark 1.4 (ii-a) and (ii-b), we have $A_0 + O_0 \subset A_0$ and $A_1 + O_1 \subset A_1$. By Lemma 1.6 and Lemma 4.3, we have the conclusion. □

Lemma 4.7 For any cut $\alpha = (A_0, A_1)$, we have $\alpha + (-\alpha) = (-\alpha) + \alpha = \iota(0)$.

Proof. Recall that $\iota(0) = (O_0, O_1)$, where $O_0 =]-\infty, 0[$ and $O_1 =]0, +\infty[$. Let c be an element of $A_0 + (-A_1)$. There exist an element $a_0 \in A_0$ and an element $a_1 \in A_1$ with $c = a_0 + (-a_1)$. Since $a_0 < a_1$, therefore $c < 0$. This means that $A_0 + (-A_1) \subset O_0$. Similarly, let c be an element of $A_1 + (-A_0)$. There exist an element $a_1 \in A_1$ and an element $a_0 \in A_0$ with $c = a_1 + (-a_0)$. Since $a_0 < a_1$, therefore $c > 0$. This means that $A_1 + (-A_0) \subset O_1$. Lemma 1.6 and Lemma 4.3 completes the proof. □

We have proved the following.

Proposition 4.8 $(X, +)$ is an abelian group. Its unit element is $\iota(0)$. The inverse of α is $-\alpha$.

5. CUTS OF $\mathbb{Q}_+$ OR $\mathbb{Q}_-$

Definition 5.1 The ordered pair $\alpha = (A_0, A_1)$ of subsets of $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) is called a *cut of* $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) if the following conditions are satisfied.

(i) $A_0 \neq \emptyset$ and $A_1 \neq \emptyset$.

(ii) $A_0 \cup A_1$ is $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) or $\mathbb{Q}_+ \setminus \{r\}$ (resp. $\mathbb{Q}_- \setminus \{r\}$) where r is a positive (resp. negative) rational number.

(iii) If $a_0 \in A_0$ and $a_1 \in A_1$, then $a_0 < a_1$ holds. Moreover, if $A_0 \cup A_1 = \mathbb{Q}_+ \setminus \{r\}$ (resp. $\mathbb{Q}_- + \setminus \{r\}$), then $a_0 < r < a_1$ holds.

(iv) A_0 does not have a maximum number. A_1 does not have a minimum number.

Lemma 5.2 (a) Assume that $\alpha = (A_0, A_1)$ is a cut of $\mathbb{Q}_+$. Put $\bar{A}_0 =]-\infty, 0] \cup A_0$ and $\bar{A}_1 = A_1$. Then, $(\bar{A}_0, \bar{A}_1)$ is a cut of $\mathbb{Q}$.

(b) Assume that $\alpha = (A_0, A_1)$ is a cut of $\mathbb{Q}_-$. Put $\bar{A}_0 = A_0$ and $\bar{A}_1 = A_1 \cup [0, +\infty[$. Then, $(\bar{A}_0, \bar{A}_1)$ is a cut of $\mathbb{Q}$.

Proof. (a) (i) Since $A_0 \subset \bar{A}_0$ and $A_1 = \bar{A}_1$, therefore $\bar{A}_0 \neq \emptyset$ and $\bar{A}_1 \neq \emptyset$.

(ii) Since $A_0 \cup A_1 = \mathbb{Q}_+$ or $\mathbb{Q}_+ \setminus \{r\}$, therefore $\bar{A}_0 \cup \bar{A}_1 =]-\infty, 0] \cup A_0 \cup A_1 = \mathbb{Q}$ or $\mathbb{Q} \setminus \{r\}$.

(iii) Since any element in $\bar{A}_0 \setminus A_0$ is smaller than any element of A_0, therefore we have the conclusion.

(iv) If $\bar{A}_0$ has a maximum number, then it must be an element of A_0. This means that A_0 has a maximum number. This is a contradiction. Since $\bar{A}_1 = A_1$, therefore it does not have a minimum number.

(b) can be proved similarly. □

Definition 5.3 In the situation in Lemma 5.2, we call $(\bar{A}_0, \bar{A}_1)$ the *extention* of $\alpha = (A_0, A_1)$.

Lemma 5.4 Assume that $\alpha = (A_0, A_1)$ is a cut of $\mathbb{Q}$.

(a) If $\alpha > \iota(0)$, then put $A'_0 = A_0 \setminus]-\infty, 0]$ and $A'_1 = A_1$. (A'_0, A'_1) is a cut of $\mathbb{Q}_+$.

(b) If $\alpha < \iota(0)$, then put $A'_0 = A_0$ and $A'_1 = A_1 \setminus [0, +\infty[$. (A'_0, A'_1) is a cut of $\mathbb{Q}_-$.

Proof. (a) (i) Since A_0 contains a positive rational number, therefore A'_0 is not empty. A'_1 is not empty because $A'_1 = A_1 \neq \emptyset$.

(ii) Since $A_0 \cup A_1 = \mathbb{Q}$ or $\mathbb{Q} \setminus \{r\}$ and $r > 0$, therefore $A'_0 \cup A'_1 = (A_0 \setminus]-\infty, 0]) \cup A_1 = (A_0 \cup A_1) \setminus]-\infty, 0] = \mathbb{Q}$ or $\mathbb{Q} \setminus \{r\}$.

(iii) For $i = 0$, 1, assume that $a_i \in A'_i$. Since an element of A'_i is an element of A_i and $\alpha = (A_0, A_1)$ is a cut of $\mathbb{Q}$, we have $a_0 < a_1$ or $a_0 < r < a_1$.

(iv) Since any element of $]-\infty, 0]$ is smaller than any element of A'_0, therefore if there exists a maximum number of A'_0, then it must be a maximum number of A_0. This contradicts to the fact that α is a cut of $\mathbb{Q}$. Therefore A'_0 does not have a maximum number. Since $\bar{A}_1 = A_1$, therefore it does not have a minimum number.

(b) can be proved similarly. □

Definition 5.5 In the situation in Lemma 5.4, we call (A'_0, A'_1) the *restriction* of α to $\mathbb{Q}_+$ or $\mathbb{Q}_-$.

Lemma 5.6 (A) Let $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ be cuts of $\mathbb{Q}_+$. Let $(\bar{A}_0, \bar{A}_1)$ and $(\bar{B}_0, \bar{B}_1)$ be their extentions. Then, (a) and (b) hold.

(a) $A_0 + B_0 = (\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$.

(b) $(A_0 + B_0, A_1 + B_1)$ is a cut of $\mathbb{Q}_+$.

(B) Let $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ be cuts of $\mathbb{Q}_-$. Let $(\bar{A}_0, \bar{A}_1)$ and $(\bar{B}_0, \bar{B}_1)$ be their extentions. Then, (a) and (b) hold.

(a) $A_1 + B_1 = (\bar{A}_1 + \bar{B}_1) \cap \mathbb{Q}_-$.

(b) $(A_0 + B_0, A_1 + B_1)$ is a cut of $\mathbb{Q}_-$.

Proof. (A)(a) Since any element of A_0 or B_0 is positive, therefore we have $A_0 + B_0 \subset (\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$.

Let c be an element of $(\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$. Note that $c > 0$ holds. There exist rational numbers a and b such that $a \in \bar{A}_0$ and $b \in \bar{B}_0$ satisfying $c = a + b$. If $a \in A_0$ and $b \in B_0$, then $c \in A_0 + B_0$ holds. If $a \notin A_0$ and $b \notin B_0$, then we have $c = a + b \leq 0$. This contradicts to the fact that $c > 0$. Assume that $a \in A_0$ and $b \notin B_0$. Since $b \leq 0$, therefore $a + b \leq a$ holds. Hence $a + b$ is an element of $\bar{A}_0$ by Remark 1.4 (ii-a). Since $a + b > 0$, therefore $a + b$ is an element of A_0. Let b' be an element of B_0. Put $b'' = \min(b', (a + b)/2)$. Since $0 < b'' < b'$ and $b' \in B_0$, therefore b'' is an element of B_0 by Remark 1.4 (ii-a). Put $a'' = a + b - b''$. Since $-b'' \geq -(a+b)/2$, therefore $a'' \geq (a+b)/2 > 0$. Since $a'' < a + b$ and $a + b \in A_0$, therefore a'' is an element of $\bar{A}_0$ by Remark 1.4 (ii-a). Hence we know that $a'' \in A_0$. Since $c = a'' + b''$, therefore $c \in A_0 + B_0$. The case that $a \notin A_0$ and $b \in B_0$ can be treated in the same way.

We have proved $A_0 + B_0 \supset (\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$.

Hence we have $A_0 + B_0 = (\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$.

(b) (i) For $i = 0$, 1, since $A_i \neq \emptyset$ and $B_i \neq \emptyset$, therefore $A_i + B_i \neq \emptyset$ holds.

(ii) In (a), we have proved that $A_0 + B_0 = (\bar{A}_0 + \bar{B}_0) \cap \mathbb{Q}_+$. Since $A_1 + B_1 = \bar{A}_1 + \bar{B}_1 \subset \mathbb{Q}_+$, therefore we have $(A_0+B_0) \cup (A_1+B_1) = ((\bar{A}_0+\bar{B}_0) \cup (\bar{A}_1+\bar{B}_1)) \cap \mathbb{Q}_+$. This means that $(A_0 + B_0) \cup (A_1 + B_1) = \mathbb{Q}_+$ or $\mathbb{Q}_+ \setminus \{r\}$, where r is the only rational number in $\mathbb{Q}_+ \setminus ((\bar{A}_0 + \bar{B}_0) \cup (\bar{A}_1 + \bar{B}_1))$.

(iii) For $i = 0$, 1, assume that $c_i \in A_i + B_i$. Since $c_i \in \bar{A}_i + \bar{B}_i$. therefore $c_0 < c_1$ or $c_0 < r < c_1$ holds.

(iv) By (a), if $A_0 + B_0$ has a maximum number, then it is a maximum number of $\bar{A}_0 + \bar{B}_0$. This contradiction shows that $A_0 + B_0$ does not have a maximum number. Since $A_1 + B_1 = \bar{A}_1 + \bar{B}_1$, therefore it does not have a minimum number.

(B) can be proved similarly. □

Definition 5.7 In the situation of Lemma 5.6, define $\alpha + \beta = (A_0 + B_0, A_1 + B_1)$.

By Lemma 5.6, we have the following.

Propositon 5.8 Let $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ be cuts of $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$). Put $\gamma = \alpha + \beta = (A_0 + A_1, B_0 + B_1)$. Let $(\bar{A}_0, \bar{A}_1)$, $(\bar{B}_0, \bar{B}_1)$ and $(\bar{C}_0, \bar{C}_1)$ be the extentions of α, β and γ respectively. Then, $(\bar{A}_0 + \bar{B}_0, \bar{A}_1 + \bar{B}_1) = (\bar{C}_0, \bar{C}_1)$ holds.

Remark. We can say that the extension and the restriction are compatible with the addition.

6. PRODUCT OF TWO CUTS OF $\mathbb{Q}_+$ OR $\mathbb{Q}_-$

Definition 6.1 Let $\alpha = (A_0, A_1)$ be a cut of $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$). Then, A_0 is called the *inner* (resp. *outer*) class of α and A_1 is called the *outer* (resp. *inner*) class of α.

Sometimes we use the notation A_{inn} (resp. A_{out}) to represent the inner (resp. outer) class of $\alpha = (A_0, A_1)$.

Remark. The class "nearer to 0" is called inner.

Let S and T be subsets of $\mathbb{Q}$. We define that $ST = \{st | s \in S, t \in T\}$.

Proposition 6.2 Let $\alpha = (A_0, A_1)$, $\beta = (B_0, B_1)$ be cuts of $\mathbb{Q}_+$ or $\mathbb{Q}_-$.

(a) If both α and β are cuts of $\mathbb{Q}_+$, then (A_0B_0, A_1B_1) is a cut of $\mathbb{Q}_+$.

(b) If both α and β are cuts of $\mathbb{Q}_-$, then (A_1B_1, A_0B_0) is a cut of $\mathbb{Q}_+$.

(c) If α is a cut of $\mathbb{Q}_+$ and β is a cut of $\mathbb{Q}_-$, then (A_1B_0, A_0B_1) is a cut of $\mathbb{Q}_-$.

(d) If α is a cut of $\mathbb{Q}_-$ and β is a cut of $\mathbb{Q}_+$, then (A_0B_1, A_1B_0) is a cut of $\mathbb{Q}_-$.

Remark. We call (A_0B_0, A_1B_1) in (a), (A_1B_1, A_0B_0) in (b), (A_1B_0, A_0B_1) in (c) and (A_0B_1, A_1B_0) in (d) the *product*s of α and β.

The inner (resp. outer) class of the product is the product of the inner (resp. outer) classes.

Proof. (a) If $a \in A_0 \cup A_1$ and $b \in B_0 \cup B_1$, then $a > 0$ and $b > 0$. Hence $ab > 0$ holds.

Assume that $x \in A_0B_0$ and a rational number y satisfies $0 < y \leq x$. There exist rational numbers a and b such that $a \in A_0$ and $b \in B_0$ with $x = ab$. Put

$b' = b(y/x)$. Since $0 < b' \leq b$ and $b \in B_0$, we have $b' \in B_0$ by Remark 1.4 (ii-a). Since $y = ab'$, therefore we have $y \in A_0B_0$.

Assume that $x \in A_1B_1$ and a rational number y satisfies $x \leq y$. There exist rational numbers a and b such that $a \in A_0$ and $b \in B_0$ with $x = ab$. Put $b' = b(y/x)$. Since $b \leq b'$ and $b \in B_1$, we have $b' \in B_1$ by Remark 1.4 (ii-b). Since $y = ab'$, therefore we have $y \in A_1B_1$.

(i) Since A_0, A_1, B_0 and B_1 are non-empty sets, therefore we have $A_0B_0 \neq \emptyset$ and $A_1B_1 \neq \emptyset$.

(ii) Assume that there exist two distinct positive rational numbers x and x' such that x, $x' \notin A_0B_0 \cup A_1B_1$. Let x'' be any rational number satisfying $x < x'' < x'$. The argument before the proof of (i) shows that $x'' \in A_0B_0$ implies $x \in A_0B_0$. This contradiction shows that $x'' \notin A_0B_0$. Similarly, we have $x'' \notin A_1B_1$. Hence we have $x'' \notin A_0B_0 \cup A_1B_1$.

Put $l = 2x'/(x + x')$. It is a rational number. Since $0 < x + x' < 2x'$, therefore we have $l > 1$. Since $(x + x')^2 > 4xx'$, therefore we have $l^2 < 4x'^2/4xx' = x'/x$. Put $k = 2l/(1 + l)$. It is a rational number. Since $2l > 1 + l$, therefore we have $k > 1$. Since $(1 + l)^2 > 2l$, therefore we have $k^2 < 4l^2/4l = l$. Hence we have $1 < k^4 < l^2 < x'/x$.

Consider the set of positive rational numbers $S = \{\cdots, k^{-3}, k^{-2}, k^{-1}, k^0, k^1, k^2, k^3, \cdots\}$. There exists an integer m such that $\{\cdots, k^{m-3}, k^{m-2}, k^{m-1}\} \subset A_0$ and that $\{k^{m+1}, k^{m+2}, k^{m+3}, \cdots\} \subset A_1$. Similarly, there exists an integer n such that $\{\cdots, k^{n-3}, k^{n-2}, k^{n-1}\} \subset B_0$ and that $\{k^{n+1}, k^{n+2}, k^{n+3}, \cdots\} \subset B_1$. Then, we have $\{\cdots, k^{m+n-4}, k^{m+n-3}, k^{m+n-2}\} \subset A_0B_0$ and $\{k^{m+n+2}, k^{m+n+3}, k^{m+n+4}, \cdots\} \subset A_1B_1$. Note that at most three element of S, namelly, k^{m+n-1}, k^{m+n}, k^{m+n+1}, are not contained in $A_0B_0 \cup A_1B_1$. Since $k^4 < x'/x$, this is a contradiction. We have proved that $A_0B_0 \cup A_1B_1 = \mathbb{Q}_+$ or $\mathbb{Q}_+ \setminus \{r\}$, where r is a positive rational number.

(iii) If $a_i \in A_i$ and $b_i \in B_i$ for $i = 0$, 1, then we have $a_0 < a_1$ and $b_0 < b_1$. Therefore we have $a_0b_0 < a_1b_1$.

Assume that $A_0B_0 \cup A_1B_1 = \mathbb{Q}_+ \setminus \{r\}$. Assumt that $c_0 \in A_0B_0$ and $c_0 \geq r$. Then we have $r \in A_0B_0$ by the argument before the proof of (i). This is a contradiction. Therefore we have $c_0 < r$. Assume that $c_1 \in A_1B_1$ and $r \geq c_1$. Then we have $r \in A_1B_1$ by the argument before the proof of (i). This is a contradiction. Therefore we have $r < c_1$.

(iv) Let c_0 be an element of A_0B_0. There exist rational element $a_0 \in A_0$ and $b_0 \in B_0$ such that $c_0 = a_0b_0$ holds. Since A_0 and B_0 do not have maximum numbers, there exist rational numbers $a'_0 \in A_0$ and $b'_0 \in B_0$ such that $a_0 < a'_0$ and $b_0 < b'_0$. Since $a_0b_0 < a'_0b'_0$, therefore c_0 is not a maximum number or A_0B_0. Hence A_0B_0 does not have a maximum number. Similarly, we can show that A_1B_1 does not have a minimum number.

The other cases (b), (c) and (d) can be proved similarly. □

7. Multiplication of cuts of $\mathbb{Q}$

Definition 7.1 Let α, β be two cuts of $\mathbb{Q}$. If one of them is equal to $\iota(0)$, then define $\alpha\beta = \iota(0)$. Otherwise, choose two cuts of $\mathbb{Q}_+$ or $\mathbb{Q}_-$ whose extensions are equal to α and β respectively and define $\alpha\beta$ to be the extension of the product of them.

Lemma 7.2 Let α and β be two cuts of $\mathbb{Q}$. Then $\alpha\beta = \beta\alpha$ holds.

Proof. If $\alpha = \iota(0)$ or $\beta = \iota(0)$, then the both sides are equal to $\iota(0)$. Otherwise, let α', β' be cuts of $\mathbb{Q}_+$ or $\mathbb{Q}_-$ whose extensions are equal to α and β respectively. Let A'_{inn} (resp. A'_{out}) be the inner (resp. outer) class of α' and let B'_{inn} (resp. B'_{out}) be the inner (resp. outer) class of β'. Since $A'_{inn}B'_{inn} = B'_{inn}A'_{inn}$ and $A'_{out}B'_{out} = B'_{out}A'_{out}$, we have the conclusion. □

Lemma 7.3 Let α, β and γ be three cuts of $\mathbb{Q}$. Then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds.

Proof. If one of them is equal to $\iota(0)$, then the both sides are equal to $\iota(0)$. Otherwise, let α', β' and γ' be cuts of $\mathbb{Q}_+$ or $\mathbb{Q}_-$ whose extensions are equal to α, β and γ respectively. Let A'_{inn} (resp. A'_{out}) be the inner (resp. outer) class of α and let B'_{inn} (resp. B'_{out}) be the inner (resp. outer) class of β and let C'_{inn} (resp. C'_{out}) be the inner (resp. outer) class of γ. Since $(A'_{inn}B'_{inn})C'_{inn} = A'_{inn}(B'_{inn}C'_{inn})$ and $(A'_{out}B'_{out})C'_{out} = A'_{out}(B'_{out}C'_{out})$, we have the conclusion. □

Lemma 7.4 The multiplication defined above commutes with the inclusion map ι, *i.e.* for any rational number r, s, we have $\iota(rs) = \iota(r)\iota(s)$.

Proof. If $r = 0$ or $s = 0$, then the both sides are equal to $\iota(0)$. Assume that $r \neq 0$ and $s \neq 0$. If $r > 0$ and $s > 0$, then $(]0, r[,]r, +\infty[)$ and $(]0, s[,]s, +\infty[)$ are cuts of $\mathbb{Q}_+$ whose extensions are equal to $\iota(r)$ and $\iota(s)$ respectively. Since $\mathbb{Q}_+ \setminus (]0, r[]0, s[\cup]r, +\infty[]s, +\infty[)$ contains rs and we know that $\iota(r)\iota(s)$ is a cut of $\mathbb{Q}_+$, therefore it must be equal to $(]0, rs[,]rs, +\infty[)$. Its extension is equal to $\iota(rs)$. Hence we have the conclusion. Other cases can be treated similarly. □

Lemma 7.5 For any cut α, we have $\alpha\iota(1) = \iota(1)\alpha = \alpha$.

Proof. We shall prove that $\alpha\iota(1) = \alpha$.

If $\alpha = \iota(0)$, then the both sides are equal to $\iota(0)$.

Assume that $\alpha > \iota(0)$. Let (A_0, A_1) be the cut of $\mathbb{Q}_+$ whose extension is equal to α. Note that $(I_0, I_1) = (]0, 1[,]1, +\infty[)$ is the cut of $\mathbb{Q}_+$ whose extension is equal to $\iota(1)$. Let a_0 be an element of A_0 and i_0 be an element of I_0. Since $0 < a_0 i_0 < a_0$ and a_0 is an element of A_0, therefore we have $a_0 i_0 \in A_0$ by Remark 1.4 (ii-a). Hence we have $A_0 I_0 \subset A_0$. Let a_1 be an element of A_1 and i_1 be an element of I_1. Since $a_1 < a_1 i_1$ and a_1 is an element of A_1, therefore we have $a_1 i_1 \in A_1$ by Remark 1.4 (ii-b). Hence we have $A_1 I_1 \subset A_1$. Apply Lemma 1.6 to the extension of $\alpha\iota(1)$ and α. Then we have $\alpha\iota(1) = \alpha$. The case $\alpha < \iota(0)$ can be proved similarly.

Lemma 7.2 shows that $\iota(1)\alpha = \alpha\iota(1) = \alpha$. □

Let S be a set of rational numbers with $0 \notin S$. Put $S^{-1} = \{s^{-1} | s \in S\}$.

Lemma 7.6 Let $\alpha = (A_0, A_1)$ be a cut of $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$). Then, (A_1^{-1}, A_0^{-1}) is a cut of $\mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) and $(A_0, A_1)(A_1^{-1}, A_0^{-1}) = \iota(1)$ holds.

Proof. (i) Since $A_0 \neq \emptyset$ and $A_1 \neq \emptyset$, therefore $A_1^{-1} \neq \emptyset$ and $A_0^{-1} \neq \emptyset$.

(ii) If $A_0 \cup A_1 = \mathbb{Q}_+$ (resp. $\mathbb{Q}_-$), then $A_1^{-1} \cup A_2^{-1} = \mathbb{Q}_+$ (resp. $\mathbb{Q}_-$). If $A_0 \cup A_1 = \mathbb{Q}_+ \setminus \{r\}$ (resp. $\mathbb{Q}_- \setminus \{r\}$) then $A_1^{-1} \cup A_2^{-1} = \mathbb{Q}_+ \setminus \{r^{-1}\}$ (resp. $\mathbb{Q}_- \setminus \{r^{-1}\}$).

(iii) If $a_0 \in A_1^{-1}$ and $a_1 \in A_0^{-1}$, then $a_0^{-1} \in A_1$ and $a_1^{-1} \in A_0$. Therefore we have $a_1^{-1} < a_0^{-1}$, hence $a_0 < a_1$ holds. If $A_0 \cup A_1 = \mathbb{Q}_+ \setminus \{r\}$ or $\mathbb{Q}_- \setminus \{r\}$, then we have $a_1^{-1} < r^{-1} < a_0^{-1}$, because $a_0 < r < a_1$.

(iv) Let a be an element of A_1^{-1}. Then, a^{-1} is an element of A_1. A_1 does not have a minimum number, there exists a rational number $a' \in A_1$ with $a' < a^{-1}$. Since $a'^{-1} \in A_1^{-1}$ and $a < a'^{-1}$. This means that A_1^{-1} does not have a maximum number. Similarly, we can show that A_0^{-1} does not have a minimum number. □

Definition 7.7 In the situation of Lemma 7.6, $(A_1^{-1}, A_0^{-1}) \in \mathbb{Q}_+$ (resp. $\mathbb{Q}_-$) is called the *inverse* of $\alpha \in \mathbb{Q}_+$ (resp. $\mathbb{Q}_-$).

Let α be a cut of $\mathbb{Q}$ which is not equal to $\iota(0)$. Define α^{-1} to be the extention of the inverse of the restriction of α. It is a cut of $\mathbb{Q}$ and is called the *inverse* of α.

Proposition 7.8 $(X \setminus \{0\}, \times)$ is an abelian group. Its unit element if $\iota(1)$. The inverse of α is α^{-1}. (The symbol "$\times$" means the mulplication.)

Lemma 7.9 Let α, β be two cuts of $\mathbb{Q}$ with $\alpha > \iota(0)$ and $\beta > \iota(0)$. Then, $-\alpha\beta = \alpha(-\beta)$ holds.

Proof. Let (A_0, B_0) and (B_0, B_1) be the restriction of α and β respectively. They are the cuts of $\mathbb{Q}_+$. Since $\alpha\beta$ is the extension (A_0B_0, A_1B_1), therefore $-\alpha\beta$ is the extention of $(-(A_1B_1), -(A_0B_0))$. Since the restriction of $-\beta$ is $(-B_1, -B_0)$, therefore $\alpha(-\beta)$ is the extension of $(A_1(-B_1), A_0(-B_0))$. Since $(-(A_1B_1), -(A_0B_0))$ is equal to $(A_1(-B_1), A_0(-B_0))$, therefore we have the conclusion. □

Lemma 7.9 Let α, β be two cuts of $\mathbb{Q}$. Then, $-\alpha\beta = \alpha(-\beta)$ holds.

Proof. If $\alpha = \iota(0)$ or $\beta = \iota(0)$, then the both sides are equal to $\iota(0)$.

Assume that $\alpha > \iota(0)$ and $\beta > \iota(0)$. Let (A_0, A_1) and (B_0, B_1) be the restriction of α and β respectively. They are cuts of $\mathbb{Q}_+$. Since $\alpha\beta$ is the extension (A_0B_0, A_1B_1), therefore $-\alpha\beta$ is the extention of $(-(A_1B_1), -(A_0B_0))$. Since the restriction of $-\beta$ is $(-B_1, -B_0)$, therefore $\alpha(-\beta)$ is the extension of $(A_1(-B_1), A_0(-B_0))$. Since $(-(A_1B_1), -(A_0B_0))$ is equal to $(A_1(-B_1), A_0(-B_0))$, therefore we have the conclusion.

Assume that $\alpha > \iota(0)$ and $\beta < \iota(0)$. Since $-\beta > 0$, therefore we have $-\alpha(-\beta) = \alpha(-(-\beta))$ by the case above. Hence we have $-\alpha\beta = \alpha(-\beta)$.

Assume that $\alpha < \iota(0)$ and $\beta < \iota(0)$. Let (A_0, A_1) and (B_0, B_1) be the restriction of α and β respectively. They are cuts of $\mathbb{Q}_-$. Since $\alpha\beta$ is the extension (A_1B_1, A_0B_0), therefore $-\alpha\beta$ is the extention of $(-(A_0B_0), -(A_1B_1))$. Since the restriction of $-\beta$ is $(-B_1, -B_0)$, therefore $\alpha(-\beta)$ is the extension of $(A_0(-B_0), A_1(-B_1))$. Since $(-(A_0B_0), -(A_1B_1))$ is equal to $(A_0(-B_0), A_1(-B_1))$, therefore we have the conclusion.

Assume that $\alpha < \iota(0)$ and $\beta > \iota(0)$. Since $-\beta < 0$, therefore we have $-\alpha(-\beta) = \alpha(-(-\beta))$ by the case above. Hence we have $-\alpha\beta = \alpha(-\beta)$. □

Lemma 7.10 Let α, β and γ be three cuts of $\mathbb{Q}$. Then $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$ hold.

Proof. First, we prove $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

If $\alpha = \iota(0)$, then the both sides are equal to $\iota(0)$.

If $\alpha \neq \iota(0)$ and $\beta + \gamma = \iota(0)$, then the left hand side is equal to $\iota(0)$. If $\beta = \gamma = \iota(0)$, then the right hand side is equal to $\iota(0)$, too.

Assume that $\alpha > \iota(0)$ and $\beta > \iota(0)$ and $\beta + \gamma = \iota(0)$. Since $\gamma = -\beta$, therefore we have $-\alpha\beta = \alpha\gamma$ by Lemma 7.9. Hence, we have $\alpha\beta + \alpha\gamma = \iota(0)$. Since $\alpha(\beta+\gamma) = \iota(0)$, therefore we have $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

Assume that $\alpha > \iota(0)$ and $\beta + \gamma > \iota(0)$. If $\beta > \iota(0)$ and $\gamma > \iota(0)$, then let (A_0, A_1), (B_0, B_1) and (C_0, C_1) be cuts of $\mathbb{Q}_+$ whose extensions are equal to α, β and γ respectively. Then, we have $A_0(B_0 + C_0) = A_0B_0 + A_0C_0$ and $A_1(B_1 + C_1) = A_1B_1 + A_1C_1$. Since the extension of $(A_0(B_0 + C_0), A_1(B_1 + C_1))$ and $(A_0B_0 + A_0C_0, A_1B_1 + A_1C_1)$ are equal to $\alpha(\beta+\gamma)$ and $\alpha\beta + \alpha\gamma$ respectively, therefore we have $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

Assume that $\alpha > \iota(0)$, $\beta > \iota(0)$, $\gamma < \iota(0)$ and $\beta+\gamma > \iota(0)$. Since $\beta-(-\gamma) > 0$ and $-\gamma > 0$, therefore we have $\alpha(\beta-(-\gamma)) + \alpha(-\gamma) = \alpha((\beta-(-\gamma))+(-\gamma)) = \alpha\beta$ by the case above. Then, we have $\alpha(\beta+\gamma) + \alpha(-\gamma) = \alpha\beta$. Since $\alpha > \iota(0)$ and $-\gamma > \iota(0)$, therefore $-\alpha(-\gamma) = \alpha\gamma$ by Lemma 7.9. Hence we have $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$.

The other cases can be treated similarly. □

Proposition 7.11 $(X, +, \times)$ is a commutative field. If we identify $r \in \mathbb{Q}$ with $\iota(r) \in X$, $(\mathbb{Q}, +, \times)$ is a subfield of $(X, +, \times)$.

8. Miscellaneous properties of X

Lemma 8.1 Let α, β and γ be three cuts of $\mathbb{Q}$. If $\alpha < \beta$, then $\alpha + \gamma < \beta + \gamma$ holds.

Proof. Put $\alpha = (A_0, A_1)$, $\beta = (B_0, B_1)$, $\gamma = (C_0, C_1)$. Since $\alpha < \beta$, therefore $A_0 \subset B_0$ holds. Since $\alpha + \gamma = (A_0 + C_0, A_0 + C_0)$ and $\beta + \gamma = (B_0 + C_0, B_1 + C_1)$ and $A_0 + C_0 \subset B_0 + C_0$, therefore we have the conclusion. □

Lemma 8.2 Let α, β and γ be three cuts of $\mathbb{Q}$. If $\alpha < \beta$ and $\gamma > \iota(0)$, then we have $\alpha\gamma < \beta\gamma$.

Proof. We have $\beta - \alpha > \iota(0)$ by Lemma 8.1. Since $\gamma > \iota(0)$, therefore we have $(\beta - \alpha)\gamma > \iota(0)$, by the definition of the product. By Lemma 7.10, we have $\beta\gamma + (-\alpha)\gamma > \iota(0)$. Hence we have $-(-\alpha)\gamma < \beta\gamma$ by Lemma 8.1. Since $-(-\alpha)\gamma = \alpha\gamma$ by Lemma 7.9, therefore we have $\alpha\gamma < \beta\gamma$. □

We have proved the following.

Proposition 8.3 $(X, +, \times)$ is a ordered field.

Proposition 8.4 The field $(X, +, \times)$ is Archimedean, *i.e.* for any two cuts α and β with $\alpha > \iota(0)$ and $\beta > \iota(0)$, there exists a natural number n such that $\iota(n)\alpha > \beta$.

Proof. Put $\alpha = (A_0, A_1)$ and $\beta = (B_0, B_1)$ and let (A'_0, A'_1) and (B'_0, B'_1) be their restrictions respectively. Choose a rational number a in A'_0 and a rational number b in B'_1. Since a and b are positive rational numbers, therefore there exists a natural number n such that $na > b$. Put $n\alpha = (C_0, C_1)$. Then, na is an element of C_0 and b is not an element of B_0. Since $na > b$, therefore b is an element of C_0 by Remark 1.4 (ii-a). Since $b \notin B_0$ and $b \in C_0$, therefore we have $B_0 \subset C_0$ and $\beta < \iota(n)\alpha$ by Proposition 2.2 (4). □

9. Completeness of real numbers

A member of X is called a *real number*. X is usually denoted by $\mathbb{R}$. For a rational number r, we identify it with $\iota(r)$.

Definition 9.1 The ordered pair $\Xi = (X_0, X_1)$ of subsets of $\mathbb{R}$ is called a *cut of* $\mathbb{R}$ if the following conditions are satified.

(i) $X_0 \neq \emptyset$ and $X_1 \neq \emptyset$.

(ii) $X_0 \cup X_1 = \mathbb{R}$ or $\mathbb{R} \setminus \{x\}$, where x is a real number.

(iii) If $x_0 \in X_0$ and $x_1 \in X_1$, then $x_0 < x_1$ holds. Moreover, if $X_0 \cup X_1 = \mathbb{R} \setminus \{x\}$, then $x_0 < x < x_1$ holds.

(iv) X_0 does not have a maximum number. X_1 does not have a minimum number.

Proposition 9.2 If $\Xi = (X_0, X_1)$ is a cut of $\mathbb{R}$, then, $X_0 \cup X_1 = \mathbb{R} \setminus \{x\}$ holds.

Proof. Put

$$B_0 = \bigcup_{(A_0, A_1) \in X_0} A_0$$

and

$$B_1 = \bigcup_{(A_0, A_1) \in X_1} A_1$$

respectively.

We shall prove that (B_0, B_1) is a cut of $\mathbb{Q}$.

(i) Since X_0 is not empty, therefore there exists a cut $(A_0, A_1) \in X_0$. Since $A_0 \neq \emptyset$, therefore B_0 is not empty. Similarly, B_1 is not empty.

(ii) Assume that r_0 is a rational number with $\iota(r_0) \in X_0$. Since X_0 does not have the maximum element, therefore there exists a cut of $\mathbb{Q}$, $(A_0, A_1) \in X_0$ such that $\iota(r_0) < (A_0, A_1)$. Since $r_0 \in A_0$, therefore $r_0 \in B_0$.

Assume that r_1 is a rational number with $\iota(r_1) \in X_1$, Since X_1 does not have the minimum element, therefore there exists a cut of $\mathbb{Q}$, $(A_0, A_1) \in X_1$ such that $\iota(r_1) > (A_0, A_1)$. Since $r_1 \in A_1$, therefore $r_1 \in B_1$.

Since (X_0, X_1) is a cut of $\mathbb{R}$, therefore $X_0 \cup X_1 = \mathbb{R}$ or $\mathbb{R} \setminus \{x\}$. If $\iota(r) \notin X_0 \cup X_1$, then $\iota(r) = x$ holds. Although we don't know whether $r \in B_i$ or not for $i = 0$, 1, we have proved that $B_0 \cup B_1 = \mathbb{Q}$ or $\mathbb{Q} \setminus \{r\}$.

(iii) If r_0 is a rational number with $r_0 \in B_0$, then there exists a cut of $\mathbb{Q}$, $(A_0, A_1) \in X_0$ such that $r_0 \in A_0$. Then, $\iota(r_0) < (A_0, A_1)$ holds. Therefore, we have $\iota(r_0) \in X_0$ by Remark 1.4 (ii-a), which is valid for a cut of $\mathbb{R}$, too.

If r_1 is a rational number with $r_1 \in B_1$, then there exists a cut of $\mathbb{Q}$, $(A_0, A_1) \in X_1$ such that $r_1 \in A_1$. Then, $\iota(r_1) > (A_0, A_1)$ holds. Therefore, we have $\iota(r_1) \in X_1$ by Remark 1.4 (ii-b), which is valid for a cut of $\mathbb{R}$, too.

If r is a rational number with $r \notin B_0 \cup B_1$, then for any $(A_0, A_1) \in X_0$ we have $\iota(r) > (A_0, A_1)$ and for any $(A_0, A_1) \in X_1$ we have $\iota(r) < (A_0, A_1)$. Therefore $\iota(r) \notin X_0 \cup X_1$.

Since (X_0, X_1) is a cut of $\mathbb{R}$, therefore $r_0 < r_1$ or $r_0 < r < r_1$ holds.

(iv) Assume that $b_0 \in B_0$. Note that b_0 is contained in some A_0 such that $(A_0, A_1) \in X_0$. Since X_0 does not have the maximum number, therefore there exists $(A'_0, A'_1) \in X_0$ with $(A'_0, A'_1) > (A_0, A_1)$. Let b be an element of $A'_0 \setminus A_0$. We have $b_0 < b$ by Remark 1.4 (ii-a"), which is valid for a cut of $\mathbb{R}$, too. Since $b \in A'_0$, therefore $b \in B_0$. Hence b_0 cannot be a maximum number of B_0.

Assume that $b_1 \in B_1$. Note that b_1 is contained in some A_1 such that $(A_0, A_1) \in X_1$. Since X_1 does not have the minimum number, therefore there exists $(A'_0, A'_1) \in X_1$ with $(A'_0, A'_1) < (A_0, A_1)$. Let b be an element of $A'_1 \setminus A_1$. We have $b < b_1$ by Remark 1.4 (ii-b"), which is valid for a cut of $\mathbb{R}$, too. Since $b \in A'_1$, threfore $b \in B_1$. Hence b_1 cannot be a minimum number of B_1.

We have proved that (B_0, B_1) is a cut of $\mathbb{Q}$.

If $(B_0, B_1) \in X_0$, then it is a maximum element of X_0, because for any element $(A_0, A_1) \in X_0$, we have $A_0 \subset B_0$.

If $(B_0, B_1) \in X_1$, then it is a minimum element of X_1, because for any element $(A_0, A_1) \in X_1$, we have $A_1 \subset B_1$.

Hence we have $(B_0, B_1) \notin X_0 \cup X_1$. We have proved that $X_0 \cup X_1 = \mathbb{R} \setminus \{x\}$, where x is equal to (B_0, B_1).

□

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