1994年京大後期理 6

(1)

$$I_{n+1}=\int_1^e(\log x)^{n+1}dx=\Big[x(\log x)^{n+1}\Big]_1^e-\int_1^e x\cdot(n+1)(\log x)^n\cdot\frac{1}{x}dx=e-(n+1)I_n$$

$$\therefore I_{n+1}=e-(n+1)I_n$$ ……(答)

(2)

$1\leqq x\leqq e$ のとき、$0\leqq\log x\leqq 1$、$0\leqq(\log x)^n\leqq 1$ であり、$(\log x)^{n+1}\leqq(\log x)^n$ であるから

$$\therefore I_n\leqq I_{n-1}\leqq\cdots\leqq I_1=\int_1^e\log x dx=\big[x\log x-x\big]_1^e=1$$

(1) より

$$I_{n+1}=e-(n+1)I_n\leqq 1\qquad e-1\leqq(n+1)I_n\qquad\therefore\frac{e-1}{n+1}\leqq I_n$$

$$I_{n+2}=e-(n+2)I_{n+1}=e-(n+2)\{e-(n+1)I_n\}=(n+1)(n+2)I_n-(n+1)e\leqq 1$$

$$(n+1)(n+2)I_n\leqq(n+1)e+1\qquad\therefore I_n\leqq\frac{(n+1)e+1}{(n+1)(n+2)}$$

以上により $\therefore\dfrac{e-1}{n+1}\leqq I_n\leqq\dfrac{(n+1)e+1}{(n+1)(n+2)}$ (証明終)