2002 年京大理 4

(1)

$$f'(x)=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}\quad\cdots\cdots(\text{答})$$

(2)

$x=\theta\cos\theta,\ y=\theta\sin\theta$ より $\quad\dfrac{dx}{d\theta}=\cos\theta-\theta\sin\theta,\ \dfrac{dy}{d\theta}=\sin\theta+\theta\cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\cos^2\theta-2\theta\sin\theta\cos\theta+\theta^2\sin^2\theta+\sin^2\theta+2\theta\sin\theta\cos\theta+\theta^2\cos^2\theta=1+\theta^2$$

求める長さを L とすると

$$L=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^{\pi}\sqrt{1+\theta^2}\,d\theta$$

$$=\left[\theta\sqrt{1+\theta^2}\right]_0^{\pi}-\int_0^{\pi}\theta\cdot\frac{\theta}{\sqrt{1+\theta^2}}\,d\theta=\pi\sqrt{1+\pi^2}-\int_0^{\pi}\frac{\theta^2}{\sqrt{1+\theta^2}}\,d\theta$$

$$=\pi\sqrt{1+\pi^2}-\int_0^{\pi}\left(\sqrt{1+\theta^2}-\frac{1}{\sqrt{1+\theta^2}}\right)d\theta=\pi\sqrt{1+\pi^2}-L+\int_0^{\pi}\frac{1}{\sqrt{1+\theta^2}}\,d\theta$$

$$2L=\pi\sqrt{1+\pi^2}+\int_0^{\pi}\frac{1}{\sqrt{1+\theta^2}}\,d\theta=\pi\sqrt{1+\pi^2}+\left[\log\left(\theta+\sqrt{1+\theta^2}\right)\right]_0^{\pi}=\pi\sqrt{1+\pi^2}+\log\left(\pi+\sqrt{1+\pi^2}\right)$$

$$\therefore L=\frac{1}{2}\pi\sqrt{1+\pi^2}+\frac{1}{2}\log\left(\pi+\sqrt{1+\pi^2}\right)\quad\cdots\cdots(\text{答})$$