2012年京大理 1

(1)

$0<a<1$ のとき　$n\to\infty$ とすると　$a^n\to 0$　$\therefore \lim_{n\to\infty}(1+a^n)^{\frac{1}{n}}=1$

$a=1$ のとき　$\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}}=\lim_{n\to\infty}2^{\frac{1}{n}}=1$

$a>1$ のとき　$(1+a^n)^{\frac{1}{n}}=\left\{a^n\left(1+\frac{1}{a^n}\right)\right\}^{\frac{1}{n}}=a\left(1+\frac{1}{a^n}\right)^{\frac{1}{n}}$　$n\to\infty$ とすると　$\frac{1}{a^n}\to 0$　$\therefore \lim_{n\to\infty}(1+a^n)^{\frac{1}{n}}=a$

以上まとめて　$0<a\leqq 1$ のとき $\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}}=1$、$1<a$ のとき $\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}}=a$　……(答)

(2)

$$\int_1^{\sqrt{3}}\frac{1}{x^2}\log\sqrt{1+x^2}\,dx=\int_1^{\sqrt{3}}\left(-\frac{1}{x}\right)'\log\sqrt{1+x^2}\,dx=\left[-\frac{1}{x}\log\sqrt{1+x^2}\right]_1^{\sqrt{3}}+\int_1^{\sqrt{3}}\frac{1}{x}\cdot\frac{1}{\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}}\,dx$$

$$=-\frac{1}{\sqrt{3}}\log 2+\log\sqrt{2}+\int_1^{\sqrt{3}}\frac{1}{1+x^2}\,dx=\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)\log 2+\int_1^{\sqrt{3}}\frac{1}{1+x^2}\,dx$$

$\int_1^{\sqrt{3}}\frac{1}{1+x^2}\,dx$ を求める。$x=\tan\theta$ とおくと　$dx=\frac{d\theta}{\cos^2\theta}$

x	$1\to\sqrt{3}$
θ	$\frac{\pi}{4}\to\frac{\pi}{3}$

$$\int_1^{\sqrt{3}}\frac{1}{1+x^2}\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{1+\tan^2\theta}\cdot\frac{d\theta}{\cos^2\theta}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta=[\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$$

$$\therefore\int_1^{\sqrt{3}}\frac{1}{x^2}\log\sqrt{1+x^2}\,dx=\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)\log 2+\frac{\pi}{12}$$　……(答)