2016 年京大理$\boxed{1}$

(1)

$$f_n(\theta)=(1+\cos\theta)\sin^{n-1}\theta$$
$$f_n'(\theta)=-\sin^n\theta+(1+\cos\theta)\cdot(n-1)\sin^{n-2}\theta\cdot(\cos\theta)=\left\{-\sin^2\theta+(n-1)\cos\theta+(n-1)\cos^2\theta\right\}\sin^{n-2}\theta$$
$$=\left\{n\cos^2\theta+(n-1)\cos\theta-1\right\}\sin^{n-2}\theta=(n\cos\theta-1)(\cos\theta+1)\sin^{n-2}\theta$$

$0\leqq\theta\leqq\dfrac{\pi}{2}$ において、$\cos\theta=\dfrac{1}{n}$ となる θ がただ 1 つ存在する。

これを $\theta=\alpha$ とすると、$f_n(\theta)$ の増減は右の通りで、$\theta=\alpha$ において極大。

$\sin\alpha=\sqrt{1-\dfrac{1}{n^2}}$ であるから、求める最大値は

$$\therefore M_n=\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)^{\frac{n-1}{2}}\ \cdots\cdots(\text{答})$$

θ	0	$\cdots$	α	$\cdots$	$\dfrac{\pi}{2}$
$f'(\theta)$		＋	0	－	
$f(\theta)$		$\nearrow$		$\searrow$	

(2)

$$(M_n)^n=\left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n^2}\right)^{\frac{n^2-n}{2}}=\frac{\left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n^2}\right)^{\frac{n^2}{2}}}{\left(1-\frac{1}{n^2}\right)^{\frac{n}{2}}}=\frac{\left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n^2}\right)^{\frac{n^2}{2}}}{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\left(1-\frac{1}{n}\right)^{\frac{n}{2}}}=\frac{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\left(\frac{n}{n-1}\right)^{\frac{n}{2}}}{\left(\frac{n^2}{n^2-1}\right)^{\frac{n^2}{2}}}$$
$$=\frac{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\left(1+\frac{1}{n-1}\right)^{\frac{n}{2}}}{\left(1+\frac{1}{n^2-1}\right)^{\frac{n^2}{2}}}=\frac{\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{2}}\left(1+\frac{1}{n-1}\right)^{\frac{1}{2}}}{\left(1+\frac{1}{n^2-1}\right)^{\frac{n^2-1}{2}}\left(1+\frac{1}{n^2-1}\right)^{\frac{1}{2}}}=\sqrt{\frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right)}{\left(1+\frac{1}{n^2-1}\right)^{n^2-1}\left(1+\frac{1}{n^2-1}\right)}}$$

$n\to\infty$ のとき、$\left(1+\dfrac{1}{n}\right)^n\to e,.\ \left(1+\dfrac{1}{n-1}\right)^{n-1}\to e,\ \left(1+\dfrac{1}{n^2-1}\right)^{n^2-1}\to e$ であるから

$$\therefore \lim_{n\to\infty}(M_n)^n=\sqrt{\frac{e\cdot e\cdot 1}{e\cdot 1}}=\sqrt{e}\ \cdots\cdots(\text{答})$$