1975年東大理 4

$a_1=\sqrt{2}=2\sin\theta_1$ より　$\sin\theta_1=\dfrac{1}{\sqrt{2}}$　$\therefore\theta_1=\dfrac{\pi}{4}$

$$a_{n+1}=2\sin\theta_{n+1}=\sqrt{2+a_n}=\sqrt{2+2\sin\theta_n}=\sqrt{2\left(1+2\sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2}\right)}=\sqrt{2\left(\sin\frac{\theta_n}{2}+\cos\frac{\theta_n}{2}\right)^2}$$

$$=\sqrt{2}\left(\sin\frac{\theta_n}{2}+\cos\frac{\theta_n}{2}\right)=2\sin\left(\frac{\theta_n}{2}+\frac{\pi}{4}\right)$$

$0<\theta_n<\dfrac{\pi}{2}$ のとき、$0<\dfrac{\theta_n}{2}+\dfrac{\pi}{4}<\dfrac{\pi}{2}$ より　$\therefore\theta_{n+1}=\dfrac{\theta_n}{2}+\dfrac{\pi}{4}$

$$\theta_{n+1}-\frac{\pi}{2}=\frac{1}{2}\left(\theta_n-\frac{\pi}{2}\right)\quad \theta_n-\frac{\pi}{2}=\left(\theta_1-\frac{\pi}{2}\right)\left(\frac{1}{2}\right)^{n-1}=-\frac{\pi}{4}\left(\frac{1}{2}\right)^{n-1}=-\frac{\pi}{2}\left(\frac{1}{2}\right)^n$$

$\therefore\theta_n=\dfrac{\pi}{2}\left\{1-\left(\dfrac{1}{2}\right)^n\right\}$ ……(答)　　$\therefore\displaystyle\lim_{n\to\infty}\theta_n=\frac{\pi}{2}$ ……(答)