1998 年東大理後期 [2]

(1)

$$\int_0^{2\pi}(\sin x - f(x))^2dx = \int_0^{2\pi}(\sin x - r\sin(x+\theta))^2dx$$

$$= \int_0^{2\pi}\sin^2 xdx - 2r\int_0^{2\pi}\sin x\sin(x+\theta)dx + r^2\int_0^{2\pi}\sin^2(x+\theta)dx = \int_0^{2\pi}\sin^2 xdx$$

$$r^2\int_0^{2\pi}\sin^2(x+\theta)dx - 2r\int_0^{2\pi}\sin x\sin(x+\theta)dx = 0$$

$r>0$ より　$\therefore r\int_0^{2\pi}\sin^2(x+\theta)dx = 2\int_0^{2\pi}\sin x\sin(x+\theta)dx$

$$\int_0^{2\pi}\sin^2(x+\theta)dx = \int_0^{2\pi}\frac{1-\cos 2(x+\theta)}{2}dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2(x+\theta)\right]_0^{2\pi} = \pi - \frac{1}{4}\sin 2\theta + \frac{1}{4}\sin 2\theta = \pi$$

$$2\int_0^{2\pi}\sin x\sin(x+\theta)dx = \int_0^{2\pi}\{\cos(x+\theta-x) - \cos(x+\theta+x)\}dx = \int_0^{2\pi}\{\cos\theta - \cos(2x+\theta)\}dx$$

$$= \left[x\cos\theta - \frac{1}{2}\sin(2x+\theta)\right]_0^{2\pi} = 2\pi\cos\theta - \frac{1}{2}\sin\theta + \frac{1}{2}\sin\theta = 2\pi\cos\theta$$

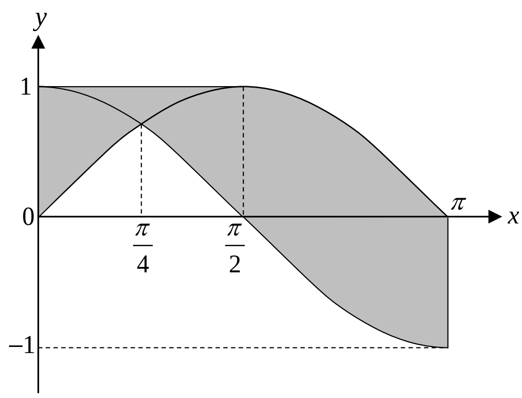
以上により　$\therefore r = \dfrac{2\pi\cos\theta}{\pi} = 2\cos\theta$　……(答)

(2)

$$f(x) = 2\cos\theta\sin(x+\theta) = \sin(x+\theta+\theta) + \sin(x+\theta-\theta) = \sin(x+2\theta) + \sin x$$

$y=\sin(x+2\theta)$ は $y=\sin x$ を x 軸方向に -2θ 移動させたものである。

$0 \leqq \theta \leqq \dfrac{\pi}{4}$ の範囲で、$y=\sin(x+2\theta)$ のグラフが動く範囲は右図の通りで、

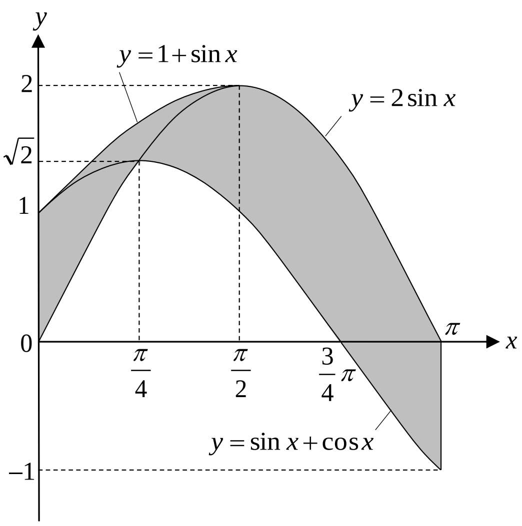


$$\begin{cases} 0 \leqq x \leqq \dfrac{\pi}{4} \text{のとき} & \sin x \leqq \sin(x+2\theta) \leqq 1 \\ \dfrac{\pi}{4} \leqq x \leqq \dfrac{\pi}{2} \text{のとき} & \cos x \leqq \sin(x+2\theta) \leqq 1 \\ \dfrac{\pi}{2} \leqq x \leqq \pi \text{のとき} & \cos x \leqq \sin(x+2\theta) \leqq \sin x \end{cases}$$

となる。これより

$$\therefore \begin{cases} 0 \leqq x \leqq \dfrac{\pi}{4} \text{のとき} & 2\sin x \leqq f(x) \leqq 1+\sin x \\ \dfrac{\pi}{4} \leqq x \leqq \dfrac{\pi}{2} \text{のとき} & \sin x + \cos x \leqq f(x) \leqq 1+\sin x \\ \dfrac{\pi}{2} \leqq x \leqq \pi \text{のとき} & \sin x + \cos x \leqq f(x) \leqq 2\sin x \end{cases}$$

$\sin x + \cos x = \sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$ であり、図示すると右図の通り。



(3)

(2) より

$$\int_0^{\frac{\pi}{4}} \left\{(1+\sin x) - 2\sin x\right\} dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\{(1+\sin x) - (\sin x + \cos x)\right\} dx + \int_{\frac{\pi}{2}}^{\pi} \left\{2\sin x - (\sin x + \cos x)\right\} dx$$

$$= \int_0^{\frac{\pi}{4}} (1-\sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\cos x) dx + \int_{\frac{\pi}{2}}^{\pi} (\sin x - \cos x) dx$$

$$= \left[x + \cos x\right]_0^{\frac{\pi}{4}} + \left[x - \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \left[-\cos x - \sin x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{4} + \frac{1}{\sqrt{2}} - 1 + \frac{\pi}{2} - 1 - \frac{\pi}{4} + \frac{1}{\sqrt{2}} + 1 + 1 = \frac{\pi}{2} + \sqrt{2}$$ ……(答)