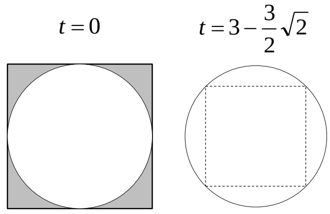
1998 年東大理 6

この四角錐を z 軸に垂直な平面で切った断面は正方形である。

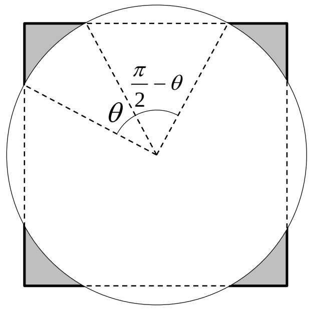


$z=t\ (0 \leqq t \leqq 3)$ における断面の一辺の長さは、$L(t)=2-\dfrac{2}{3}t$ と表される。

$L(t) \leqq \sqrt{2}$ のとき、$x^2+y^2 \geqq 1$ の部分は存在しないから、

$2-\dfrac{2}{3}t \geqq \sqrt{2} \quad 0 \leqq t \leqq 3-\dfrac{3}{2}\sqrt{2}$ ——①

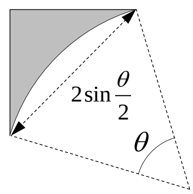
①の範囲の $z=t$ における断面のうち、$x^2+y^2 \geqq 1$ の部分を考える。



図において $0 \leqq \theta \leqq \dfrac{\pi}{2}$ であり、$x^2+y^2 \geqq 1$ の部分の面積 $S(t)$ を θ で表すと

$$S(t)=4\left(\sin^2\frac{\theta}{2}-\frac{1}{2}\theta+\frac{1}{2}\sin\theta\right)=4\left(\frac{1-\cos\theta}{2}-\frac{1}{2}\theta+\frac{1}{2}\sin\theta\right)$$
$$=2(1-\theta-\cos\theta+\sin\theta)$$

$L(t)$ を θ で表すと



$$\frac{1}{2}L(t)=\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{\sqrt{2}}{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)=1-\frac{1}{3}t$$

$$\frac{\sqrt{2}}{4}\left(-\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)d\theta=-\frac{1}{3}dt \quad \therefore dt=-\frac{3\sqrt{2}}{4}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta$$

求める体積は

$$\int_0^{3-\frac{3}{2}\sqrt{2}} S(t)dt$$
$$=\int_{\frac{\pi}{2}}^{0} 2(1-\theta-\cos\theta+\sin\theta)\left\{-\frac{3\sqrt{2}}{4}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\right\}d\theta=\frac{3\sqrt{2}}{2}\int_0^{\frac{\pi}{2}}(1-\theta-\cos\theta+\sin\theta)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta$$

ここで

$$(1-\theta-\cos\theta+\sin\theta)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)$$
$$=\cos\frac{\theta}{2}-\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}+\theta\sin\frac{\theta}{2}-\cos\theta\cos\frac{\theta}{2}+\cos\theta\sin\frac{\theta}{2}+\sin\theta\cos\frac{\theta}{2}-\sin\theta\sin\frac{\theta}{2}$$
$$=\cos\frac{\theta}{2}-\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}+\theta\sin\frac{\theta}{2}-\left(\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}\right)+\left(\sin\theta\cos\frac{\theta}{2}+\cos\theta\sin\frac{\theta}{2}\right)$$
$$=\cos\frac{\theta}{2}-\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}+\theta\sin\frac{\theta}{2}-\cos\left(\theta-\frac{\theta}{2}\right)+\sin\left(\theta+\frac{\theta}{2}\right)$$
$$=\sin\frac{3}{2}\theta-\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}+\theta\sin\frac{\theta}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin\frac{3}{2}\theta d\theta = \left[-\frac{2}{3}\cos\frac{3}{2}\theta\right]_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{3} + \frac{2}{3} \qquad \int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2} d\theta = \left[-2\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = -\sqrt{2} + 2$$

$$\int_0^{\frac{\pi}{2}} \theta\cos\frac{\theta}{2} d\theta = \int_0^{\frac{\pi}{2}} \theta\left(2\sin\frac{\theta}{2}\right)' d\theta = \left[2\theta\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2} d\theta = \frac{\pi}{\sqrt{2}} - 2(2-\sqrt{2}) = 2\sqrt{2} - 4 + \frac{\sqrt{2}}{2}\pi$$

$$\int_0^{\frac{\pi}{2}} \theta\sin\frac{\theta}{2} d\theta = \int_0^{\frac{\pi}{2}} \theta\left(-2\cos\frac{\theta}{2}\right)' d\theta = \left[-2\theta\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2} d\theta = -\frac{\pi}{\sqrt{2}} + 2\left[2\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = 2\sqrt{2} - \frac{\sqrt{2}}{2}\pi$$

以上により

$$\frac{3\sqrt{2}}{2}\left\{\left(\frac{\sqrt{2}}{3} + \frac{2}{3}\right) - (2-\sqrt{2}) - \left(2\sqrt{2} - 4 + \frac{\sqrt{2}}{2}\pi\right) + \left(2\sqrt{2} - \frac{\sqrt{2}}{2}\pi\right)\right\} = \frac{3\sqrt{2}}{2}\left(\frac{4\sqrt{2}}{3} + \frac{8}{3} - \sqrt{2}\pi\right)$$

$$= 4 + 4\sqrt{2} - 3\pi \quad \cdots\cdots(\text{答})$$

※2012 年東工大 [6] では、三角錐について類題が出題された。