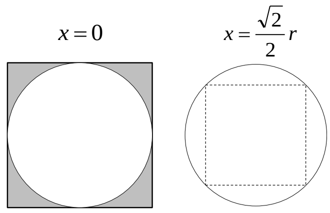
2005 年東大理 6

$x^2+y^2 \leqq r^2,\ z^2+x^2 \leqq r^2$ が表す立体は、直交する円柱の共通部分であり、
x 軸に垂直な平面で切った断面は正方形である。

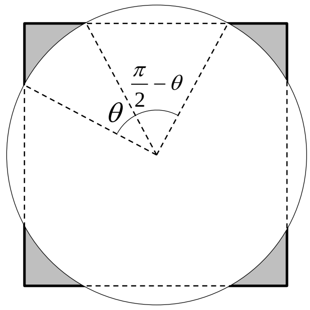


$x=r\sin\phi \left(-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2}\right)$ における断面の一辺の長さは　$L(x)=2r\cos\phi$

$2r\cos\phi \leqq \sqrt{2}r$ のとき、$y^2+z^2 \geqq r^2$ の部分は存在しないから、

$\cos\phi \geqq \frac{\sqrt{2}}{2} \quad -\frac{\pi}{4} \leqq \phi \leqq \frac{\pi}{4}$ ——①

①の範囲の $x=r\sin\phi$ における断面のうち、$y^2+z^2 \geqq r^2$ の部分を考える。

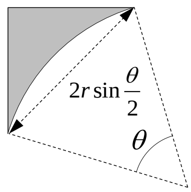


図において $0 \leqq \theta \leqq \frac{\pi}{2}$ であり、$y^2+z^2 \geqq r^2$ の部分の面積 $S(x)$ を θ で表すと

$$S(x)=4r^2\left(\sin^2\frac{\theta}{2}-\frac{1}{2}\theta+\frac{1}{2}\sin\theta\right)=4r^2\left(\frac{1-\cos\theta}{2}-\frac{1}{2}\theta+\frac{1}{2}\sin\theta\right)$$
$$=2r^2(1-\theta-\cos\theta+\sin\theta)$$

$L(x)$ を θ で表すと

$$\frac{1}{2}L(x)=r\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=r\cos\phi$$



対称性から、$0 \leqq \phi \leqq \frac{\pi}{4},\ 0 \leqq \theta \leqq \frac{\pi}{2}$ のとき $\phi=\frac{\pi}{4}-\frac{\theta}{2}$ と表せて、

$$x=r\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right) \quad \therefore dx=-\frac{1}{2}r\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)d\theta=-\frac{\sqrt{2}}{4}r\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)d\theta$$

求める体積は

$$2\int_0^{\frac{\sqrt{2}}{2}r} S(x)dx$$
$$=2\int_{\frac{\pi}{2}}^{0} 2r^2(1-\theta-\cos\theta+\sin\theta)\left\{-\frac{\sqrt{2}}{4}r\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\right\}d\theta=\sqrt{2}r^3\int_0^{\frac{\pi}{2}}(1-\theta-\cos\theta+\sin\theta)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)d\theta$$

ここで

$$(1-\theta-\cos\theta+\sin\theta)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$$
$$=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}-\theta\sin\frac{\theta}{2}-\cos\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}+\sin\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}$$
$$=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}-\theta\sin\frac{\theta}{2}-\left(\cos\theta\cos\frac{\theta}{2}-\sin\theta\sin\frac{\theta}{2}\right)+\left(\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}\right)$$
$$=\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-\theta\cos\frac{\theta}{2}-\theta\sin\frac{\theta}{2}-\cos\left(\theta+\frac{\theta}{2}\right)+\sin\left(\theta-\frac{\theta}{2}\right)$$
$$=\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}-\cos\frac{3}{2}\theta-\theta\cos\frac{\theta}{2}-\theta\sin\frac{\theta}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2}\,d\theta = \left[2\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = \sqrt{2} \qquad \int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2}\,d\theta = \left[-2\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = -\sqrt{2}+2 \qquad \int_0^{\frac{\pi}{2}} \cos\frac{3}{2}\theta\,d\theta = \left[\frac{2}{3}\sin\frac{3}{2}\theta\right]_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{3}$$

$$\int_0^{\frac{\pi}{2}} \theta\cos\frac{\theta}{2}\,d\theta = \int_0^{\frac{\pi}{2}} \theta\left(2\sin\frac{\theta}{2}\right)'d\theta = \left[2\theta\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2}\,d\theta = \frac{\pi}{\sqrt{2}} - 2(2-\sqrt{2}) = 2\sqrt{2} - 4 + \frac{\sqrt{2}}{2}\pi$$

$$\int_0^{\frac{\pi}{2}} \theta\sin\frac{\theta}{2}\,d\theta = \int_0^{\frac{\pi}{2}} \theta\left(-2\cos\frac{\theta}{2}\right)'d\theta = \left[-2\theta\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \cos\frac{\theta}{2}\,d\theta = -\frac{\pi}{\sqrt{2}} + 2\left[2\sin\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = 2\sqrt{2} - \frac{\sqrt{2}}{2}\pi$$

以上により

$$\sqrt{2}r^3\left\{\sqrt{2} + 2(2-\sqrt{2}) - \frac{\sqrt{2}}{3} - \left(2\sqrt{2} - 4 + \frac{\sqrt{2}}{2}\pi\right) - \left(2\sqrt{2} - \frac{\sqrt{2}}{2}\pi\right)\right\} = \sqrt{2}r^3\left(8 - \frac{16}{3}\sqrt{2}\right)$$

$$= 8\left(\sqrt{2} - \frac{4}{3}\right)r^3 \quad \cdots\cdots(答)$$

※1998 年理系 6 に酷似。