2011 年東大理 3

(1)

点 P は半径 t の円周上にあり、 $P(t\cos\theta,\ t\sin\theta)$ とすると、 $t\theta = L$ であるから

$$\therefore u(t) = t\cos\frac{L}{t},\ v(t) = t\sin\frac{L}{t} \quad \cdots\cdots(\text{答})$$

(2)

$$u'(t) = \cos\frac{L}{t} + t\cdot\left(-\sin\frac{L}{t}\right)\cdot\left(-\frac{L}{t^2}\right) = \cos\frac{L}{t} + \frac{L}{t}\sin\frac{L}{t} \qquad v'(t) = \sin\frac{L}{t} + t\cdot\cos\frac{L}{t}\cdot\left(-\frac{L}{t^2}\right) = \sin\frac{L}{t} - \frac{L}{t}\cos\frac{L}{t}$$

$$\{u'(t)\}^2 + \{v'(t)\}^2 = 1 + \frac{L^2}{t^2} \qquad \sqrt{\{u'(t)\}^2 + \{v'(t)\}^2} = \frac{\sqrt{t^2+L^2}}{t} \qquad \therefore f(a) = \int_a^1 \frac{\sqrt{t^2+L^2}}{t}\,dt$$

$x = \sqrt{t^2+L^2}$ と置くと

$$x^2 = t^2 + L^2 \qquad 2x\,dx = 2t\,dt \qquad \frac{x}{t^2}dx = \frac{1}{t}dt \qquad \therefore \frac{x}{x^2 - L^2}dx = \frac{1}{t}dt$$

$$f(a) = \int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}} \frac{x^2}{x^2-L^2}dx = \int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}} \left\{1 + \frac{L^2}{(x+L)(x-L)}\right\}dx = \int_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}} \left\{1 + \frac{L}{2}\left(\frac{1}{x-L} - \frac{1}{x+L}\right)\right\}dx$$

$$= \left[x - \frac{L}{2}\{\log(x+L) - \log(x-L)\}\right]_{\sqrt{a^2+L^2}}^{\sqrt{1+L^2}} = \sqrt{1+L^2} - \sqrt{a^2+L^2} - \frac{L}{2}\log\frac{\sqrt{1+L^2}+L}{\sqrt{1+L^2}-L} + \frac{L}{2}\log\frac{\sqrt{a^2+L^2}+L}{\sqrt{a^2+L^2}-L}$$

$$= \sqrt{1+L^2} - \sqrt{a^2+L^2} - \frac{L}{2}\log\frac{(\sqrt{1+L^2}+L)^2}{(1+L^2)-L^2} + \frac{L}{2}\log\frac{(\sqrt{a^2+L^2}+L)^2}{(a^2+L^2)-L^2}$$

$$= \sqrt{1+L^2} - \sqrt{a^2+L^2} - L\log(\sqrt{1+L^2}+L) + L\log(\sqrt{a^2+L^2}+L) - L\log a \quad \cdots\cdots(\text{答})$$

(3)

$$\frac{f(a)}{\log a} = \frac{\sqrt{1+L^2} - \sqrt{a^2+L^2} - L\log(\sqrt{1+L^2}+L) + L\log(\sqrt{a^2+L^2}+L)}{\log a} - L$$

$\lim_{a\to+0}\log a = -\infty$ であるから $\quad \therefore \lim_{a\to+0}\dfrac{f(a)}{\log a} = -L \quad \cdots\cdots(\text{答})$