2019年東大理 1

$$\int_0^1 \left(x^2+\frac{x}{\sqrt{1+x^2}}\right)\left(1+\frac{x}{(1+x^2)\sqrt{1+x^2}}\right)dx$$

$$=\int_0^1 x^2dx+\int_0^1 \frac{x}{\sqrt{1+x^2}}dx+\int_0^1 \frac{x^3}{(1+x^2)^{3/2}}dx+\int_0^1 \frac{x^2}{(1+x^2)^2}dx$$

$$\int_0^1 x^2dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3} \qquad \int_0^1 \frac{x}{\sqrt{1+x^2}}dx=\left[\sqrt{1+x^2}\right]_0^1=\sqrt{2}-1$$

$x=\tan\theta$ とおくと　$dx=\dfrac{d\theta}{\cos^2\theta}$

x	$0\to 1$
θ	$0\to \frac{\pi}{4}$

$$\int_0^1 \frac{x^3}{(1+x^2)^{3/2}}dx=\int_0^{\frac{\pi}{4}}\tan^3\theta\cos^3\theta\cdot\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\frac{\sin^3\theta}{\cos^2\theta}d\theta=\int_0^{\frac{\pi}{4}}\frac{1-\cos^2\theta}{\cos^2\theta}\cdot\sin\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{\sin\theta}{\cos^2\theta}-\sin\theta\right)d\theta=\left[\frac{1}{\cos\theta}+\cos\theta\right]_0^{\frac{\pi}{4}}=\sqrt{2}+\frac{\sqrt{2}}{2}-2=\frac{3}{2}\sqrt{2}-2$$

同様に

$$\int_0^1 \frac{x^2}{(1+x^2)^2}dx=\int_0^{\frac{\pi}{4}}\tan^2\theta\cos^4\theta\cdot\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}}\sin^2\theta\,d\theta=\int_0^{\frac{\pi}{4}}\frac{1-\cos 2\theta}{2}d\theta$$

$$=\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}-\frac{1}{4}$$

以上により、求める値は

$$\frac{1}{3}+\sqrt{2}-1+\frac{3}{2}\sqrt{2}-2+\frac{\pi}{8}-\frac{1}{4}=\frac{\pi}{8}+\frac{5}{2}\sqrt{2}-\frac{35}{12}$$ ……(答)