

2019 年東大理 [1]

$$\begin{aligned} & \int_0^1 \left(x^2 + \frac{x}{\sqrt{1+x^2}} \right) \left(1 + \frac{x}{(1+x^2)\sqrt{1+x^2}} \right) dx \\ &= \int_0^1 x^2 dx + \int_0^1 \frac{x}{\sqrt{1+x^2}} dx + \int_0^1 \frac{x^3}{(1+x^2)^{3/2}} dx + \int_0^1 \frac{x^2}{(1+x^2)^2} dx \end{aligned}$$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} \quad \int_0^1 \frac{x}{\sqrt{1+x^2}} dx = \left[\sqrt{1+x^2} \right]_0^1 = \sqrt{2} - 1$$

$$x = \tan \theta \text{ とおくと } dx = \frac{d\theta}{\cos^2 \theta} \quad \begin{array}{l} x \mid 0 \rightarrow 1 \\ \theta \mid 0 \rightarrow \frac{\pi}{4} \end{array}$$

$$\begin{aligned} \int_0^1 \frac{x^3}{(1+x^2)^{3/2}} dx &= \int_0^{\frac{\pi}{4}} \tan^3 \theta \cos^3 \theta \cdot \frac{d\theta}{\cos^2 \theta} = \int_0^{\frac{\pi}{4}} \frac{\sin^3 \theta}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{1 - \cos^2 \theta}{\cos^2 \theta} \cdot \sin \theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \left(\frac{\sin \theta}{\cos^2 \theta} - \sin \theta \right) d\theta = \left[\frac{1}{\cos \theta} + \cos \theta \right]_0^{\frac{\pi}{4}} = \sqrt{2} + \frac{\sqrt{2}}{2} - 2 = \frac{3}{2}\sqrt{2} - 2 \end{aligned}$$

同様に

$$\begin{aligned} \int_0^1 \frac{x^2}{(1+x^2)^2} dx &= \int_0^{\frac{\pi}{4}} \tan^2 \theta \cos^4 \theta \cdot \frac{d\theta}{\cos^2 \theta} = \int_0^{\frac{\pi}{4}} \sin^2 \theta d\theta = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} d\theta \\ &= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

以上により、求める値は

$$\frac{1}{3} + \sqrt{2} - 1 + \frac{3}{2}\sqrt{2} - 2 + \frac{\pi}{8} - \frac{1}{4} = \frac{\pi}{8} + \frac{5}{2}\sqrt{2} - \frac{35}{12} \dots\dots (\text{答})$$