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(1)

$t = x^2$とすると　$x = \sqrt{t}$　$dx = \frac{1}{2\sqrt{t}}dt$　$A_k = \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} |\sin(x^2)|dx = \frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{\sqrt{t}}dt$

$k\pi \leqq t \leqq (k+1)\pi$の範囲で、$\frac{1}{\sqrt{(k+1)\pi}} \leqq \frac{1}{\sqrt{t}} \leqq \frac{1}{\sqrt{k\pi}}$が成り立つから、

$$\frac{|\sin t|}{\sqrt{(k+1)\pi}} \leqq \frac{|\sin t|}{\sqrt{t}} \leqq \frac{|\sin t|}{\sqrt{k\pi}} \quad \frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{\sqrt{(k+1)\pi}}dt \leqq \frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{\sqrt{t}}dt \leqq \frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{\sqrt{k\pi}}dt$$

$$\therefore \frac{1}{2\sqrt{(k+1)\pi}}\int_{k\pi}^{(k+1)\pi} |\sin t|dt \leqq A_k \leqq \frac{1}{2\sqrt{k\pi}}\int_{k\pi}^{(k+1)\pi} |\sin t|dt$$

正弦関数の対称性より、$\int_{k\pi}^{(k+1)\pi} |\sin t|dt = \int_0^{\pi} \sin t\, dt = [-\cos t]_0^{\pi} = 1 + 1 = 2$ であるから

$$\therefore \frac{1}{\sqrt{(k+1)\pi}} \leqq A_k \leqq \frac{1}{\sqrt{k\pi}}$$ (証明終)

(2)

(1)で示した式より　$\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{(k+1)\pi}} \leqq \frac{1}{\sqrt{n}}\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} |\sin(x^2)|dx \leqq \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{k\pi}}$ ——①

①の各辺の、$k = n$から$k = 2n - 1$までの和をとると

$$\frac{1}{\sqrt{n}}\sum_{k=n}^{2n-1} \frac{1}{\sqrt{(k+1)\pi}} \leqq \frac{1}{\sqrt{n}}\int_{\sqrt{n\pi}}^{\sqrt{2n\pi}} |\sin(x^2)|dx \leqq \frac{1}{\sqrt{n}}\sum_{k=n}^{2n-1} \frac{1}{\sqrt{k\pi}}$$

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \frac{1}{\sqrt{(n+k)\pi}} \leqq B_n \leqq \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \frac{1}{\sqrt{(n+k)\pi}} \quad \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\pi}} \leqq B_n \leqq \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\pi}}$$

区分求積法より

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\pi}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sqrt{\left(1+\frac{k}{n}\right)\pi}} = \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1+x}}dx = \frac{1}{\sqrt{\pi}}\left[2\sqrt{1+x}\right]_0^1 = \frac{2\sqrt{2}-2}{\sqrt{\pi}}$$

はさみうちの原理より　$\therefore \lim_{n\to\infty} B_n = \frac{2\sqrt{2}-2}{\sqrt{\pi}}$ ……(答)