FAST CONSTRUCTION OF REAL NUMBERS BY HALF-CUTS

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Abstract: A fast construction of the real numbers by using half-cuts of the dense subset of $\mathbb{Q}_+$ which is based on the idea of Dedekind's cut. We can naturally define the addition and the multiplication of the positive real numbers and show the completeness of the real numbers.

1. Half-Cuts of a Dense Subset of $\mathbb{Q}_+$

Let $\mathbb{Q}_+$ be the set of all the positive rational numbers.

Definition 1.1 Let I be an interval of $\mathbb{Q}_+ \cup \{0\}$. A subset S of $\mathbb{Q}_+$ is called *dense* in I if $a, b \in I$ with $a < b$, then there is an element $x \in S$ such that $a < x < b$.

Let $\mathbb{S}_+$ be a dense subset of $\mathbb{Q}_+$ such that it is closed in respect to the addition and the multiplication. For exapmle, the set of all the positive finite binaries, the set of all the positive finite decimals, or $\mathbb{Q}_+$ itself.

Definition 1.2 A subset A of $\mathbb{S}_+$ is called a *half-cut* of $\mathbb{S}_+$ (or just a *half-cut*) if the following conditions are satisfied. (i) A is not empty. (ii) A has an *upper bound* $M \in \mathbb{S}_+$, *i.e.* if $a \in A$ then $a \leq M$. (iii) If $a \in A$ and $a' \in \mathbb{S}_+$ with $a' < a$, then $a' \in A$. (iv) A does not have a maximum number.

The set of all the half-cuts of $\mathbb{S}_+$ is denoted by X_+.

Definition 1.3 For two half-cuts A and B, define $A \leq B$ if $A \subset B$. Define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 1.4 The relation $\leq$ is a total order on X_+, *i.e.* for half-cuts A, B and C, we have (i) $A \leq A$, (ii) if $A \leq B$ and $B \leq A$ then $A = B$, (iii) if $A \leq B$ and $B \leq C$ then $A \leq C$, (iv) $A \leq B$ or $B \leq A$.

Proof. (i), (ii), (iii) are clear. (iv) Assume that $A \not\leq B$. There is a number r with $r \in A$ and $r \notin B$. Since A is a half-cut, therefore if $s \in \mathbb{S}_+$ and $s < r$ then $s \in A$. Since B is a half-cut, therefore if $s \in B$ then $s < r$. Hence $B \leq A$ holds. □

Definition 1.5 Define the map $\iota : \mathbb{S}_+ \to X_+$ by $\iota(r) = \{a \in \mathbb{S}_+ | a < r\}$.

Fact 1.6 The map ι preserves order, *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Lemma 1.7 Assume that r, s and $c \in \mathbb{S}_+$ with $c < r + s$ (*resp.* $c < rs$). Then, there exist r', $s' \in \mathbb{S}_+$ with $r' < r$, $s' < s$, and $c \leq r' + s'$ (*resp.* $c \leq r's'$).

Proof. Since $\iota(r)$ is dense in $[0, r[$ and $\iota(s)$ is dense in $[0, s[$, therefore $\{r' + s' | r' \in \iota(r), s' \in \iota(s)\}$ (*resp.* $\{r's' | r' \in \iota(r), s' \in \iota(s)\}$) is dense in $[0, r + s[$ (*resp.* $[0, rs[$).

2. Addition and Multiplication of Half-Cuts

Definition 2.1 For half-cuts A and B, put $A + B = \{x \in \mathbb{S}_+ |$ There exist $a \in A,\ b \in B$ such that $x \leq a + b\}$ and put $AB = \{x \in \mathbb{S}_+ |$ There exist $a \in A,\ b \in B$ such that $x \leq ab\}$.

Lemma 2.2 If A and B are half-cuts, then $C = A + B$ (*resp.* $C = AB$) is a half-cut.

Proof. (i) Since A and B are non-empty, therefore C is not empty. (ii) Since A and B have upper bounds, therefore C has an upper bound. (iii) Clear by definiton.

(iv) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). There exists $a' \in A$ with $a < a'$. Then, $c' = a' + b$ (*resp.* $c' = a'b$) is an element of C with $c < c'$. □

Fact 2.3 Let A, B and C be half-cuts. Then $A+B = B+A$, $(A+B)+C = A+(B+C)$, $AB = BA$, $(AB)C = A(BC)$, $A(B+C) = AB+AC$, $(A+B)C = AC+BC$.

proof. We shall prove $(A + B) + C \subset A + (B + C)$ and $A(B + C) \subset AB + AC$.

Assume that $x \in (A + B) + C$. There exist $a \in A$, $b \in B$, $y \in A + B$, and $c \in C$ such that $y \leq a + b$, $x \leq y + c$. Therefore $x \leq a + b + c$. Since $a \in A$ and $b + c \in B + C$, therefore we have $x \leq a + (b + c) \in A + (B + C)$.

If $x \in AB + AC$, then there exist a, $a' \in A$, $b \in B$, and $c \in C$ such that $x \leq ab + a'c$. If $a < a'$, then $x' = a'b + a'c = a'(b + c) \in A(B + C)$ and $x < x'$. □

Lemma 2.4 Let A, B and C be half-cuts of $\mathbb{S}_+$. If $A < B$, then $A + C < B + C$ (*resp.* $AC < BC$).

Proof. Assume that r, $r + \delta \in B \setminus A$ and $\delta > 0$ (*resp.* r, $r\rho \in B \setminus A$ and $\rho > 1$). There exists an element $s \in C$ with $s + \delta \notin C$ (*resp.* $s\rho \notin C$). Note that $(r + \delta) + s \in B + C$ (*resp.* $(r\rho)s \in BC$) and $r + (\delta + s) \notin A + C$ (*resp.* $r(\rho s) \notin AC$). Because if there exist $r' \in A$ and $s' \in C$ with $(r + \delta) + s = r' + s'$ (*resp.* $(r\rho)s = r's'$), then $r < r'$ or $s < s'$ must hold. This is a contradiction. □

Lemma 2.5 The addition and multiplication defined above commute with the inclusion map ι, *i.e.* if r, $s \in \mathbb{S}_+$, then $\iota(r + s) = \iota(r) + \iota(s)$ and $\iota(rs) = \iota(r)\iota(s)$.

Proof. $\iota(r + s) \supset \iota(r) + \iota(s)$ (*resp.* $\iota(rs) \supset \iota(r)\iota(s)$) is clear. Assume that $x \in \iota(r + s)$, *i.e.* $x < r + s$ (*resp.* $x \in \iota(rs)$, *i.e.* $x < rs$). Apply Lemma 1.7. □

Lemma 2.6 For any half-cut A, we have $A\iota(1) = \iota(1)A = A$.

Proof. $A\iota(1) \subset A$ is clear. Assume that $a \in A$. There exists an element $a' \in A$ with $a < a'$. We have $a < a'1$. Apply Lemma1.7. □

Lemma 2.7 The multiplication on X_+ is Archimedean, *i.e.* for any two half-cuts A and B, there exists a natural number n such that $A < \iota(n)B$.

Proof. Let M be an upper bound of A and let b be an element of B. Since $\mathbb{Q}_+$ is Archimedean, there exists a natural number n with $M < nb$. □

3. Completeness of Positive Real Numbers

See [1].

4. Constructing Real Numbers from $\mathbb{R}_+$

See [1].

References

[1] Rapid construction of real numbers by half-cuts (preprint).

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