RAPID CONSTRUCTION OF REAL NUMBERS BY HALF-CUTS

IWASE, ZJUÑICI

Abstract: A rapid construction of the real numbers by using half-cuts of the positive rational numbers which is based on the idea of Dedekind's cut. We can naturally define the addition and the multiplication of the positive real numbers and show the completeness of the positive real numbers.

1. Half-Cuts of $\mathbb{Q}_+$

Let $\mathbb{Q}_+$ be the set of all the positive rational numbers.

Definition 1.1 A subset A of $\mathbb{Q}_+$ is called a *half-cut* of $\mathbb{Q}_+$ (or just a *half-cut*) if the following conditions are satisfied. (i) A is not empty. (ii) A has an *upper bound* $M \in \mathbb{Q}_+$, *i.e.* if $a \in A$ then $a \leq M$. (iii) If $a \in A$ and $a' \in \mathbb{Q}_+$ with $a' < a$, then $a' \in A$. (iv) A does not have a maximum number.

The set of all the half-cuts of $\mathbb{Q}_+$ is denoted by X_+.

Definition 1.2 For two half-cuts A and B, define $A \leq B$ if $A \subset B$. Define $A < B$ if $A \leq B$ and $A \neq B$.

Lemma 1.3 The relation $\leq$ is a total order on X_+, *i.e.* for half-cuts A, B and C, we have (i) $A \leq A$, (ii) if $A \leq B$ and $B \leq A$ then $A = B$, (iii) if $A \leq B$ and $B \leq C$ then $A \leq C$, (iv) $A \leq B$ or $B \leq A$.

Proof. (i), (ii), (iii) are clear. (iv) Assume that $A \not\leq B$. There is an element r with $r \in A$ and $r \notin B$. Since A is a half-cut, therefore if $s \in \mathbb{Q}_+$ and $s < r$ then $s \in A$. Since B is a half-cut, therefore if $s \in B$ then $s < r$. Hence $B \leq A$ holds. □

Definition 1.4 Define the map $\iota : \mathbb{Q}_+ \to X_+$ by $\iota(r) = \{a \in \mathbb{Q}_+ | a < r\}$.

Fact 1.5 The map ι preserves order, *i.e.* if $r < s$, then $\iota(r) < \iota(s)$ holds.

Lemma 1.6 Assume that r, s and $c \in \mathbb{Q}_+$ with $c < r + s$ (*resp.* $c < rs$). Then, there exist r', $s' \in \mathbb{Q}_+$ with $r' < r$, $s' < s$, and $c = r' + s'$ (*resp.* $c = r's'$).

Proof. Find $r' \in \mathbb{Q}_+$ with $c - s < r' < r$ (*resp.* $c/s < r' < r$). □

2. Addition and Multiplication of Half-Cuts of $\mathbb{Q}_+$

Definition 2.1 For half-cuts A and B, put $A + B = \{a + b \mid a \in A,\ b \in B\}$ and put $AB = \{ab \mid a \in A,\ b \in B\}$.

Lemma 2.2 If A and B are half-cuts, then $C = A + B$ (*resp.* $C = AB$) is a half-cut.

Proof. (i) Since A and B are non-empty, therefore C is not empty. (ii) Since A and B have upper bounds, therefore C has an upper bound. (iii) See Lemma 1.6. (iv) Assume that $c \in C$. There exist $a \in A$ and $b \in B$ such that $c = a + b$ (*resp.* $c = ab$). There exists $a' \in A$ with $a < a'$. Then, $c' = a' + b$ (*resp.* $c' = a'b$) is an element of C with $c < c'$. □

Fact 2.3 Let A, B and C be half-cuts. Then $A+B = B+A$, $(A+B)+C = A+(B+C)$, $AB = BA$, $(AB)C = A(BC)$, $A(B+C) = AB+AC$, $(A+B)C = AC+BC$.

Proof. We shall prove $A(B+C) \supset AB+AC$. If $x \in AB+AC$, then there exist a, $a' \in A$, $b \in B$, and $c \in C$ such that $x = ab + a'c$. If $a < a'$, then $x' = a'b + a'c = a'(b+c) \in A(B+C)$ and $x < x'$. □

Lemma 2.4 Let A, B and C be half-cuts. If $A < B$, then $A+C < B+C$ (*resp.* $AC < BC$).

Proof. Assume that r, $r+\delta \in B \setminus A$ and $\delta > 0$ (*resp.* r, $r\rho \in B \setminus A$ and $\rho > 1$). There exists an element $s \in C$ with $s + \delta \notin C$ (*resp.* $s\rho \notin C$). Note that $(r+\delta)+s \in B+C$ (*resp.* $(r\rho)s \in BC$). If $(r+\delta)+s \in A+C$ (*resp.* $(r\rho)s \in AC$), then there exist $r' \in A$ and $s' \in C$ such that $(r+\delta)+s = r'+s'$ (*resp.* $r\rho s = r's'$). Since $r' < r$ and $s' < s+\delta$ (*resp.* $s' < s\rho$), therefore this is a contradiction. □

Lemma 2.5 The addition and multiplication defined above commute with the inclusion map ι, *i.e.* if r, $s \in \mathbb{Q}_+$, then $\iota(r+s) = \iota(r)+\iota(s)$ and $\iota(rs) = \iota(r)\iota(s)$.

Proof. $\iota(r+s) \supset \iota(r)+\iota(s)$ (*resp.* $\iota(rs) \supset \iota(r)\iota(s)$) is clear. Assume that $x \in \iota(r+s)$, *i.e.* $x < r+s$ (*resp.* $x \in \iota(rs)$, *i.e.* $x < rs$). Apply Lemma 1.6. □

Lemma 2.6 For any half-cut A, we have $A\iota(1) = \iota(1)A = A$.

Proof. $A\iota(1) \subset A$ is clear. Assume that $a \in A$. There exists an element $a' \in A$ with $a < a'$. We have $a < a'1$. Apply Lemma1.6. □

Lemma 2.7 The multiplication on X_+ is Archimedean, *i.e.* for any two half-cuts A and B, there exists a natural number n such that $A < \iota(n)B$.

Proof. Let M be an upper bound of A and let b be an element of B. Since $\mathbb{Q}_+$ is Archimedean, therefore there exists a natural number n with $M < nb$. □

3. Completeness of Positive Real Numbers

A member of X_+ is called a positive *real number*. Denote X_+ by $\mathbb{R}_+$.

Let Λ be a non-empty subset of $\mathbb{R}_+$. A positive real number α is called a *supremum* of Λ if it satisfies the following conditions. (i) If $A \in \Lambda$, then $A \leq \alpha$. (ii) If $\alpha' < \alpha$ and $\alpha' \in \mathbb{R}_+$, then there exists $A \in \Lambda$ with $\alpha' < A$.

Let Λ be a non-empty subset of $\mathbb{R}_+$ and M be an element of $\mathbb{R}_+$. We call M an *upper bound* of Λ if $A \in \Lambda$ implies $A \leq M$.

Propositon 3.1 Let Λ be a non-empty subsets of $\mathbb{R}_+$ with an upper bound. Then $\cup\Lambda$ is a supremum of Λ.

Proof. The set $\alpha = \cup\Lambda$ is a half-cut. The conditions (i) and (ii) are clear. (iii) If $a \in \cup\Lambda$ and $a' \in \mathbb{Q}_+$ with $a' < a$, then $a' \in \cup\Lambda$ holds. In fact, there exists $A \in \Lambda$ with $a \in A$. Then, $a' \in A$. Hence $a' \in \cup\Lambda$. (iv) $\cup\Lambda$ does not have a maximum number. In fact, if $a \in \cup\Lambda$, then there exists $A \in \Lambda$ with $a \in A$. Since A is a half-cut, therefore there exists $a' \in A \subset \cup\Lambda$ with $a < a'$.

We shall prove that α is a supremum of Λ. If $A \in \Lambda$, then $A \leq \alpha$. If $\alpha' < \alpha$, then there exists $r \in \alpha$ with $r \notin \alpha'$. There exists $A \in \Lambda$ with $r \in A$. Since $\alpha' < A$, we have the conclusion. □

4. Appendix. Constructing $\mathbb{R}$ from $\mathbb{R}_+$

If A, $C \in \mathbb{R}_+$ with $A < C$ (*resp.* [sic]), then there exists one and only one element $B \in \mathbb{R}_+$ with $A + B = C$ (*resp.* $AB = C$). In fact, let B be the supremum of the set of all the half-cuts B' with $A + B' < C$ (*resp.* $AB' < C$).

Hence $\mathbb{R}$ is $(-\mathbb{R}_+) \cup \{0\} \cup \mathbb{R}_+$.

Departments of the School of Mathematics and Physics, College of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa, 920-1192, Japan.

E-mail address: iwase@staff.kanazawa-u.ac.jp