1973年京大理 4

$$\frac{(\sqrt{n(n+1)}-n)^3}{n}-\frac{(\sqrt{n(n+1)}-(n+1))^3}{n+1}$$

$$=\frac{(\sqrt{n^2+n}-n)(n^2+n-2n\sqrt{n^2+n}+n^2)}{n}-\frac{(\sqrt{n^2+n}-(n+1))(n^2+n-2(n+1)\sqrt{n^2+n}+(n+1)^2)}{n+1}$$

$$=(\sqrt{n^2+n}-n)(2n+1-2\sqrt{n^2+n})-(\sqrt{n^2+n}-(n+1))(2n+1-2\sqrt{n^2+n})$$

$$=2n+1-2\sqrt{n^2+n}=\frac{(2n+1)^2-4(n^2+n)}{2n+1+2\sqrt{n^2+n}}=\frac{1}{2n+1+2\sqrt{n^2+n}}$$

$$n\left[\frac{(\sqrt{n(n+1)}-n)^3}{n}-\frac{(\sqrt{n(n+1)}-(n+1))^3}{n+1}\right]=\frac{n}{2n+1+2\sqrt{n^2+n}}=\frac{1}{2+\frac{1}{n}+2\sqrt{1+\frac{1}{n}}}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{(\sqrt{n(n+1)}-n)^3}{n}-\frac{(\sqrt{n(n+1)}-(n+1))^3}{n+1}\right]=\frac{1}{4}$$ ……(答)