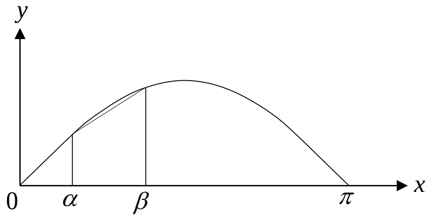
1997 年京大理 $\boxed{4}$

(1)



$\sin x$ の凸性より　$\displaystyle\int_\alpha^\beta \sin x dx > \frac{1}{2}(\beta-\alpha)(\sin\alpha+\sin\beta)$

対称性より　$\displaystyle\int_\alpha^\beta \sin x dx + \int_{\pi-\beta}^{\pi-\alpha} \sin x dx = 2\int_\alpha^\beta \sin x dx > (\beta-\alpha)(\sin\alpha+\sin\beta)$

$\sin\beta=\sin(\pi-\beta)$ であるから　$\displaystyle\therefore \int_\alpha^\beta \sin x dx + \int_{\pi-\beta}^{\pi-\alpha} \sin x dx > (\beta-\alpha)(\sin\alpha+\sin(\pi-\beta))$　(証明終)

(2)

$$\int_\alpha^\beta \sin x dx + \int_{\pi-\beta}^{\pi-\alpha} \sin x dx = 2\int_\alpha^\beta \sin x dx = 2\left[-\cos x\right]_\alpha^\beta = 2(\cos\alpha-\cos\beta)$$

(1) より

$$2(\cos\alpha-\cos\beta) > (\beta-\alpha)(\sin\alpha+\sin(\pi-\beta)) \quad \therefore \sin\alpha+\sin(\pi-\beta) < \frac{2}{\beta-\alpha}(\cos\alpha-\cos\beta)$$

$\alpha=0,\ \beta=\dfrac{\pi}{8}$ とすると　$\displaystyle\sin\frac{7\pi}{8} < \frac{16}{\pi}\left(1-\cos\frac{\pi}{8}\right)$ ——①

$\alpha=\dfrac{\pi}{8},\ \beta=\dfrac{2\pi}{8}$ とすると　$\displaystyle\sin\frac{\pi}{8}+\sin\frac{6\pi}{8} < \frac{16}{\pi}\left(\cos\frac{\pi}{8}-\cos\frac{2\pi}{8}\right) = \frac{16}{\pi}\left(\cos\frac{\pi}{8}-\frac{1}{\sqrt{2}}\right)$ ——②

$\alpha=\dfrac{2\pi}{8},\ \beta=\dfrac{3\pi}{8}$ とすると　$\displaystyle\sin\frac{2\pi}{8}+\sin\frac{5\pi}{8} < \frac{16}{\pi}\left(\cos\frac{2\pi}{8}-\cos\frac{3\pi}{8}\right) = \frac{16}{\pi}\left(\frac{1}{\sqrt{2}}-\cos\frac{3\pi}{8}\right)$ ——③

$\alpha=\dfrac{3\pi}{8},\ \beta=\dfrac{4\pi}{8}$ とすると　$\displaystyle\sin\frac{3\pi}{8}+\sin\frac{4\pi}{8} < \frac{16}{\pi}\left(\cos\frac{3\pi}{8}-\cos\frac{4\pi}{8}\right) = \frac{16}{\pi}\cos\frac{3\pi}{8}$ ——④

①～④を辺々足すと　$\displaystyle\therefore \sum_{k=1}^{7}\sin\frac{k\pi}{8} < \frac{16}{\pi}$　(証明終)