2000年京大理 5

(1)

$$c_{n+2}=(n+3)\int_0^1 x^{n+2}\cos\pi x dx=\frac{n+3}{\pi}\int_0^1 x^{n+2}(\sin\pi x)'dx=\frac{n+3}{\pi}\left\{\left[x^{n+2}\sin\pi x\right]_0^1-(n+2)\int_0^1 x^{n+1}\sin\pi x dx\right\}$$

$$=-\frac{(n+3)(n+2)}{\pi}\int_0^1 x^{n+1}\sin\pi x dx=\frac{(n+3)(n+2)}{\pi^2}\int_0^1 x^{n+1}(\cos\pi x)'dx$$

$$=\frac{(n+3)(n+2)}{\pi^2}\left\{\left[x^{n+1}\cos\pi x\right]_0^1-(n+1)\int_0^1 x^n\cos\pi x dx\right\}=-\frac{(n+3)(n+2)}{\pi^2}(1+c_n)$$

$$1+c_n=-\frac{\pi^2}{(n+3)(n+2)}c_{n+2}\qquad \therefore c_n=-\frac{\pi^2}{(n+3)(n+2)}c_{n+2}-1$$ ……(答)

(2)

(1)より $c_n=-\dfrac{\pi^2}{n+2}\displaystyle\int_0^1 x^{n+2}\cos\pi x dx-1$

ここで、$0\leqq x\leqq 1$のとき　$-1\leqq\cos\pi x\leqq 1$　$-x^{n+2}\leqq x^{n+2}\cos\pi x\leqq x^{n+2}$

$$\therefore -\int_0^1 x^{n+2}dx\leqq\int_0^1 x^{n+2}\cos\pi x dx\leqq\int_0^1 x^{n+2}dx$$

$\displaystyle\int_0^1 x^{n+2}dx=\left[\frac{x^{n+3}}{n+3}\right]_0^1=\frac{1}{n+3}$ より　$\therefore -\dfrac{\pi^2}{(n+2)(n+3)}-1\leqq c_n\leqq\dfrac{\pi^2}{(n+2)(n+3)}-1$

はさみうちの原理により　$\therefore \lim_{n\to\infty}c_n=-1$ ……(答)

(3)

$c_n-c=c_n+1=-\dfrac{\pi^2}{(n+3)(n+2)}c_{n+2}$ であるから　$\dfrac{c_{n+1}-c}{c_n-c}=\dfrac{(n+3)(n+2)}{(n+4)(n+3)}\cdot\dfrac{c_{n+2}}{c_{n+3}}=\dfrac{n+2}{n+4}\cdot\dfrac{c_{n+2}}{c_{n+3}}$

$$\therefore \lim_{n\to\infty}\frac{c_{n+1}-c}{c_n-c}=\lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{4}{n}}\cdot\frac{-1}{-1}=1$$ ……(答)