2008 年京大文 1

$$\{f(x)\}^2=(ax^2+bx+c)^2=a^2x^4+b^2x^2+c^2+2abx^3+2bcx+2cax^2=a^2x^4+2abx^3+(b^2+2ca)x^2+2bcx+c^2$$

$$\int_{-1}^{1}\{f(x)\}^2dx=2\int_0^1\left\{a^2x^4+(b^2+2ca)x^2+c^2\right\}dx=2\left[\frac{a^2}{5}x^5+\frac{b^2+2ca}{3}x^3+c^2x\right]_0^1=\frac{2}{5}a^2+\frac{2}{3}b^2+2c^2+\frac{4}{3}ca$$

$$\{f'(x)\}^2=(2ax+b)^2=4a^2x^2+4abx+b^2$$

$$(1-x^2)\{f'(x)\}^2=(1-x^2)(4a^2x^2+4abx+b^2)=4a^2x^2+4abx+b^2-4a^2x^4-4abx^3-b^2x^2$$

$$=-4a^2x^4-4abx^3+(4a^2-b^2)x^2+4abx+b^2$$

$$\int_{-1}^{1}(1-x^2)\{f'(x)\}^2dx=2\int_0^1\left\{-4a^2x^4+(4a^2-b^2)x^2+b^2\right\}dx=2\left[-\frac{4}{5}a^2x^5+\frac{4a^2-b^2}{3}x^3+b^2x\right]_0^1$$

$$=-\frac{8}{5}a^2+\frac{8}{3}a^2-\frac{2}{3}b^2+2b^2=\frac{16}{15}a^2+\frac{4}{3}b^2$$

以上により

$$6\int_{-1}^{1}\{f(x)\}^2dx-\int_{-1}^{1}(1-x^2)\{f'(x)\}^2dx$$

$$=\frac{12}{5}a^2+4b^2+12c^2+8ca-\frac{16}{15}a^2-\frac{4}{3}b^2=\frac{4}{3}a^2+\frac{8}{3}b^2+12c^2+8ca=\frac{4}{3}(a+3c)^2+\frac{8}{3}b^2\geqq 0$$

$$\therefore\int_{-1}^{1}(1-x^2)\{f'(x)\}^2dx\leqq 6\int_{-1}^{1}\{f(x)\}^2dx$$ （証明終）

等号は、$a+3c=0$ かつ $b=0$ のとき成立。