2021年京大理 4

$y=\log(1+\cos x)$ より　$\dfrac{dy}{dx}=\dfrac{-\sin x}{1+\cos x}$

$$1+\left(\frac{dy}{dx}\right)^2=\frac{(1+\cos x)^2+\sin^2 x}{(1+\cos x)^2}=\frac{2(1+\cos x)}{(1+\cos x)^2}=\frac{2}{1+\cos x}=\frac{1}{\cos^2\frac{x}{2}}$$

求める長さは　$L=\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^{\frac{\pi}{2}}\frac{1}{\cos\frac{x}{2}}dx$　$t=\dfrac{x}{2}$ と置き換えると

$$L=2\int_0^{\frac{\pi}{4}}\frac{1}{\cos t}dt=2\int_0^{\frac{\pi}{4}}\frac{\cos t}{\cos^2 t}dt=2\int_0^{\frac{\pi}{4}}\frac{\cos t}{1-\sin^2 t}dt=\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\sin t}+\frac{1}{1-\sin t}\right)\cos t\,dt$$

$$=[\log(1+\sin t)-\log(1-\sin t)]_0^{\frac{\pi}{4}}=\log\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}=\log\frac{\sqrt{2}+1}{\sqrt{2}-1}=\log\left(\sqrt{2}+1\right)^2=2\log\left(\sqrt{2}+1\right)$$ ……(答)