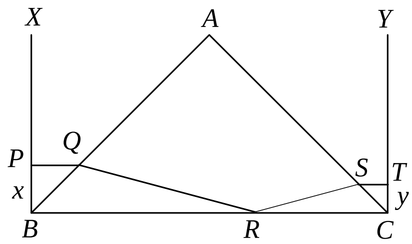
1962年東大理$\boxed{2}$文$\boxed{2}$共通



$PQ/\!/BC$ より $PQ\perp BX$ で、$BP=PQ=x$、$\angle BQP=\dfrac{\pi}{4}$ であるから、

$$\cos\angle BQP=\frac{1}{\sqrt{2}}\quad \cos\angle AQR=\frac{1}{\sqrt{2}}\cos\angle BQP=\frac{1}{2}\quad \therefore \angle AQR=\frac{\pi}{3}$$

$$\angle BRQ=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}\quad \therefore \angle CRS=\frac{\pi}{12}$$

$\triangle BRQ \backsim \triangle CRS$ は明らかで、$\angle ASR=\dfrac{\pi}{3}\quad \cos\angle ASR=\dfrac{1}{2}\quad \cos\angle CST=\sqrt{2}\cos\angle ASR=\dfrac{1}{\sqrt{2}}$

$\angle CST=\dfrac{\pi}{4}$ より、$ST/\!/BC$ となる。

$BP=x$ のとき、BR を x で表すことを考える。

$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$QR\sin\dfrac{\pi}{12}=x$ より　$QR=\dfrac{x}{\sin\frac{\pi}{12}}=\dfrac{4x}{\sqrt{6}-\sqrt{2}}=(\sqrt{6}+\sqrt{2})x$

$$\therefore BR=x+QR\cos\frac{\pi}{12}=x+(\sqrt{6}+\sqrt{2})x\cdot\frac{\sqrt{6}+\sqrt{2}}{4}=\left\{1+\frac{(\sqrt{6}+\sqrt{2})^2}{4}\right\}x=(3+\sqrt{3})x$$

ただし、$BR<BC=2\sqrt{2}$ より、$(3+\sqrt{3})x<2\sqrt{2}\quad x<\dfrac{2\sqrt{2}}{3+\sqrt{3}}=\dfrac{2\sqrt{2}(3-\sqrt{3})}{6}=\sqrt{2}-\dfrac{\sqrt{6}}{3}$ ——①

①の条件下で、$RC=2\sqrt{2}-BR=2\sqrt{2}-(3+\sqrt{3})x\quad BR:RC=x:y$ であるから

$$\therefore x:y=(3+\sqrt{3})x:\left\{2\sqrt{2}-(3+\sqrt{3})x\right\}\quad (3+\sqrt{3})xy=x\left\{2\sqrt{2}-(3+\sqrt{3})x\right\}$$

$$y=\frac{2\sqrt{2}}{3+\sqrt{3}}-x=\sqrt{2}-\frac{\sqrt{6}}{3}-y\quad \therefore x+y=\sqrt{2}-\frac{\sqrt{6}}{3}\quad \cdots\cdots(答)$$

ただし、$0<x<\sqrt{2}-\dfrac{\sqrt{6}}{3},\ 0<y<\sqrt{2}-\dfrac{\sqrt{6}}{3}$ である。