

(1)

$$P\left(a, \frac{1}{a}\right) (a > 0), Q\left(-t, -\frac{1}{t}\right) (t > 0) \text{ とすると } PQ^2 = (a+t)^2 + \left(\frac{1}{a} + \frac{1}{t}\right)^2 = (a+t)^2 \left(1 + \frac{1}{a^2 t^2}\right)$$

$$f(t) = (a+t)^2 \left(1 + \frac{1}{a^2 t^2}\right) \text{ とすると}$$

$$\begin{aligned} f'(t) &= 2(a+t) \left(1 + \frac{1}{a^2 t^2}\right) + (a+t)^2 \left(-\frac{2}{a^2 t^3}\right) = 2(a+t) \left(1 + \frac{1}{a^2 t^2} - \frac{a+t}{a^2 t^3}\right) = \frac{2(a+t)(a^2 t^3 + t - a - t)}{a^2 t^3} \\ &= \frac{2(a+t)(a^2 t^3 + t - a - t)}{a^2 t^3} = \frac{2a(a+t)(at^3 - 1)}{a^2 t^3} \end{aligned}$$

$f(t)$ の増減は右の通りで、 $t = a^{-\frac{1}{3}}$ のとき最小。

t	0	...	$a^{-\frac{1}{3}}$...
$f'(t)$		-	0	+
$f(t)$		↘		↗

$$\begin{aligned} f\left(a^{-\frac{1}{3}}\right) &= \left(a + a^{-\frac{1}{3}}\right)^2 \left(1 + \frac{1}{a^{\frac{4}{3}}}\right) = \left(\frac{a^{\frac{4}{3}} + 1}{a^{\frac{1}{3}}}\right)^2 \frac{(a^{\frac{4}{3}} + 1)}{a^{\frac{4}{3}}} = \frac{(a^{\frac{4}{3}} + 1)^3}{a^2} \\ &= \left(\frac{a^{\frac{4}{3}} + 1}{a^{\frac{2}{3}}}\right)^3 = \left(a^{\frac{2}{3}} + a^{-\frac{2}{3}}\right)^3 \end{aligned}$$

PQ の最小値は $\left(a^{\frac{2}{3}} + a^{-\frac{2}{3}}\right)^{\frac{3}{2}}$ (答)

(2)

$$t = a^{-\frac{1}{3}} \text{ のとき、} QP \text{ の傾きは } \frac{\frac{1}{a} + a^{\frac{1}{3}}}{a + a^{-\frac{1}{3}}} = \frac{1 + a^{\frac{4}{3}}}{a^2 + a^{\frac{2}{3}}} = a^{-\frac{2}{3}}$$

これが $\tan 30^\circ = \frac{1}{\sqrt{3}}$ に等しいから $a^{-\frac{2}{3}} = \frac{1}{\sqrt{3}}$ $a^{\frac{2}{3}} = \sqrt{3}$ $\therefore a = 3^{\frac{3}{4}}$ (答)