1973年東大理 $\boxed{5}$

$$\frac{1}{1+x^2}-\frac{t}{(1+t^2)x}=\frac{(1+t^2)x-t(1+x^2)}{(1+t^2)(1+x^2)x}=-\frac{tx^2-(1+t^2)x+t}{(1+t^2)(1+x^2)x}=-\frac{t}{(1+t^2)(1+x^2)x}\left(x-\frac{1}{t}\right)(x-t)$$

$\dfrac{1}{1+x^2} \geqq \dfrac{t}{(1+t^2)x}$ を満たす $x$ の範囲は、$\dfrac{1}{t} \leqq x \leqq t$ であるから

$$f(t)=\int_{\frac{1}{t}}^{t}\left\{\frac{1}{1+x^2}-\frac{t}{(1+t^2)x}\right\}dx=\int_{\frac{1}{t}}^{t}\frac{1}{1+x^2}\,dx-\frac{t}{1+t^2}\int_{\frac{1}{t}}^{t}\frac{1}{x}\,dx=\int_{\frac{1}{t}}^{t}\frac{1}{1+x^2}\,dx-\frac{t}{1+t^2}\left[\log x\right]_{\frac{1}{t}}^{t}$$

$$=\int_{\frac{1}{t}}^{t}\frac{1}{1+x^2}\,dx-\frac{t}{1+t^2}\left(\log t-\log\frac{1}{t}\right)=\int_{\frac{1}{t}}^{t}\frac{1}{1+x^2}\,dx-\frac{2t\log t}{1+t^2}$$

$$f'(t)=\frac{1}{1+t^2}-\frac{1}{1+\frac{1}{t^2}}\cdot\left(-\frac{1}{t^2}\right)-2\cdot\frac{(\log t+1)(1+t^2)-t\log t\cdot 2t}{(1+t^2)^2}=\frac{2}{1+t^2}-2\cdot\frac{(\log t+1)(1+t^2)-2t^2\log t}{(1+t^2)^2}$$

$$=\frac{2}{1+t^2}-2\cdot\frac{\log t+t^2\log t+1+t^2-2t^2\log t}{(1+t^2)^2}=\frac{2}{1+t^2}-\frac{2}{1+t^2}+\frac{2(t^2-1)\log t}{(1+t^2)^2}$$

$$=\frac{2(t^2-1)\log t}{(t^2+1)^2}$$ ……(答)