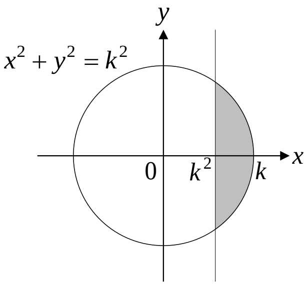
1994 年東大理 3

(1)

この立体の $z=k$ における断面は右図の通り。



$S(k)=2\int_{k^2}^{k}\sqrt{k^2-x^2}\,dx$ であり、$x=k\cos t$ とおくと　$dx=-k\sin t\,dt$

x	$k^2\to k$
t	$\theta\to 0$

$$S(k)=2\int_{\theta}^{0}\sqrt{k^2(1-\cos^2 t)}\cdot(-k\sin t\,dt)=2k^2\int_0^{\theta}\sin^2 t\,dt=\cos^2\theta\int_0^{\theta}(1-\cos 2t)dt$$

$$=\cos^2\theta\left[t-\frac{1}{2}\sin 2t\right]_0^{\theta}=\theta\cos^2\theta-\frac{1}{2}\cos^2\theta\sin 2\theta=\theta\cos^2\theta-\cos^3\theta\sin\theta$$ ……(答)

(2)

$k=\cos\theta$ より　$dk=-\sin\theta\,d\theta$

k	$0\to 1$
θ	$\frac{\pi}{2}\to 0$

$$V=\int_0^1 S(k)dk=\int_{\frac{\pi}{2}}^{0}(\theta\cos^2\theta-\cos^3\theta\sin\theta)\cdot(-\sin\theta\,d\theta)=\int_0^{\frac{\pi}{2}}(\theta\cos^2\theta\sin\theta-\cos^3\theta\sin^2\theta)d\theta$$

$$\int_0^{\frac{\pi}{2}}\theta\cos^2\theta\sin\theta\,d\theta=\int_0^{\frac{\pi}{2}}\theta\left(-\frac{1}{3}\cos^3\theta\right)'d\theta=\left[-\frac{1}{3}\theta\cos^3\theta\right]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)\cos\theta\,d\theta=\frac{1}{3}\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_0^{\frac{\pi}{2}}=\frac{2}{9}$$

$$\int_0^{\frac{\pi}{2}}\cos^3\theta\sin^2\theta\,d\theta=\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)\sin^2\theta\cos\theta\,d\theta=\int_0^{\frac{\pi}{2}}(\sin^2\theta-\sin^4\theta)\cos\theta\,d\theta$$

$$=\left[\frac{1}{3}\sin^3\theta-\frac{1}{5}\sin^5\theta\right]_0^{\frac{\pi}{2}}=\frac{1}{3}-\frac{1}{5}=\frac{2}{15}$$

$$\therefore V=\frac{2}{9}-\frac{2}{15}=\frac{10-6}{45}=\frac{4}{45}$$ ……(答)