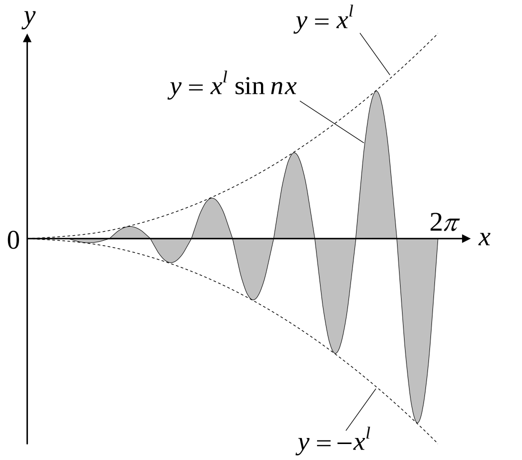
2000年東大理後期 2

(1)



$$V_n=\pi\int_0^{2\pi}y^2dx=\pi\int_0^{2\pi}x^{2l}\sin^2 nxdx=\frac{\pi}{2}\int_0^{2\pi}x^{2l}(1-\cos 2nx)dx$$
$$=\frac{\pi}{2}\left(\int_0^{2\pi}x^{2l}dx-\int_0^{2\pi}x^{2l}\cos 2nxdx\right)$$

ここで、

$$\int_0^{2\pi}x^{2l}\cos 2nxdx=\int_0^{2\pi}x^{2l}\left(\frac{\sin 2nx}{2n}\right)'dx$$
$$=\left[x^{2l}\cdot\frac{\sin 2nx}{2n}\right]_0^{2\pi}-\frac{l}{n}\int_0^{2\pi}x^{2l-1}\sin 2nxdx=-\frac{l}{n}\int_0^{2\pi}x^{2l-1}\sin 2nxdx$$

$-1\leqq\sin 2nx\leqq 1$ であるから、$x\geqq 0$ のとき $-x^{2l-1}\leqq x^{2l-1}\sin 2nx\leqq x^{2l-1}$

$$\therefore -\int_0^{2\pi}x^{2l-1}dx\leqq\int_0^{2\pi}x^{2l-1}\sin 2nxdx\leqq\int_0^{2\pi}x^{2l-1}dx$$

$\int_0^{2\pi}x^{2l-1}dx=\left[\frac{x^{2l}}{2l}\right]_0^{2\pi}=\frac{2^{2l-1}\pi^{2l}}{l}$ より $\therefore -\frac{2^{2l-1}\pi^{2l}}{n}\leqq\frac{l}{n}\int_0^{2\pi}x^{2l-1}\sin 2nxdx\leqq\frac{2^{2l-1}\pi^{2l}}{n}$

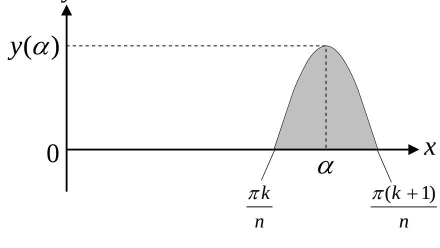
$\lim_{n\to\infty}\frac{2^{2l-1}\pi^{2l}}{n}=0$ であるから、はさみうちの原理により $\therefore \lim_{n\to\infty}\frac{l}{n}\int_0^{2\pi}x^{2l-1}\sin 2nxdx=0$

$$\therefore \lim_{n\to\infty}V_n=\frac{\pi}{2}\int_0^{2\pi}x^{2l}dx=\frac{\pi}{2}\left[\frac{x^{2l+1}}{2l+1}\right]_0^{2\pi}=\frac{2^{2l}\pi^{2l+2}}{2l+1}\quad\cdots\cdots(答)$$

(2)

$y=x^l|\sin nx|, 0\leqq x\leqq 2\pi$ のグラフと x 軸で囲まれる図形を D_n とすると、W_n は D_n を y 軸の回りに回転させてできる回転体の体積に等しい。k を $0\leqq k\leqq 2n-1$ なる整数としたとき、D_n の $\frac{\pi k}{n}\leqq x\leqq\frac{\pi(k+1)}{n}$ の部分を y 軸の回りに回転させてできる回転体の体積を w_k とする。この区間で、$y=x^l|\sin nx|$ は $x=\alpha$ で極大値をとるとし、

x を $\frac{\pi k}{n}\leqq x\leqq\alpha$ のとき $x_1(y)$、$\alpha\leqq x\leqq\frac{\pi(k+1)}{n}$ のとき $x_2(y)$ と表すと



$$w_k=\pi\int_0^{y(\alpha)}{x_2}^2dy-\pi\int_0^{y(\alpha)}{x_1}^2dy=\pi\int_{\frac{\pi(k+1)}{n}}^{\alpha}{x_2}^2y'dx_2-\pi\int_{\frac{\pi k}{n}}^{\alpha}{x_1}^2y'dx_1$$

$$=\pi\int_{\frac{\pi(k+1)}{n}}^{\alpha}{x_2}^2y'dx_2+\pi\int_{\alpha}^{\frac{\pi k}{n}}{x_1}^2y'dx_1=\pi\int_{\frac{\pi(k+1)}{n}}^{\frac{\pi k}{n}}x^2y'dx=\pi\left(\left[x^2y\right]_{\frac{\pi(k+1)}{n}}^{\frac{\pi k}{n}}-2\int_{\frac{\pi(k+1)}{n}}^{\frac{\pi k}{n}}xydx\right)$$

$\dfrac{\pi k}{n}\leqq x\leqq\dfrac{\pi(k+1)}{n}$ のとき、 $y=(-1)^k\cdot x^l\sin nx$ であるから

$$w_k=2\pi\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}xydx=(-1)^k\cdot2\pi\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^{l+1}\sin nxdx=(-1)^k\cdot2\pi\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^{l+1}\left(-\frac{\cos nx}{n}\right)'dx$$

$$=(-1)^k\cdot2\pi\left(\left[-x^{l+1}\cdot\frac{\cos nx}{n}\right]_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}+\frac{l+1}{n}\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx\right)$$

$$=(-1)^k\cdot2\pi\left\{-\frac{\pi^{l+1}(k+1)^{l+1}}{n^{l+2}}\cdot(-1)^{k+1}+\frac{\pi^{l+1}k^{l+1}}{n^{l+2}}\cdot(-1)^k+\frac{l+1}{n}\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx\right\}$$

$$=2\pi^{l+2}\cdot\frac{1}{n}\left\{\left(\frac{k+1}{n}\right)^{l+1}+\left(\frac{k}{n}\right)^{l+1}\right\}+(-1)^k\cdot2\pi\cdot\frac{l+1}{n}\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx$$

$$\therefore W_n=\sum_{k=0}^{2n-1}w_k=2\pi^{l+2}\cdot\frac{1}{n}\sum_{k=0}^{2n-1}\left\{\left(\frac{k+1}{n}\right)^{l+1}+\left(\frac{k}{n}\right)^{l+1}\right\}+2\pi\cdot\frac{l+1}{n}\sum_{k=0}^{2n-1}(-1)^k\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx$$

ここで、$-1\leqq\cos nx\leqq1$ であるから、$x\geqq0$ のとき　$-x^l\leqq x^l\cos nx\leqq x^l$

$$-\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^ldx\leqq\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx\leqq\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^ldx\qquad-\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^ldx\leqq(-1)^k\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx\leqq\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^ldx$$

$\displaystyle\sum_{k=0}^{2n-1}\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^ldx=\int_0^{2\pi}x^ldx=\left[\frac{x^{l+1}}{l+1}\right]_0^{2\pi}=\frac{2^{l+1}\pi^{l+1}}{l+1}$ より　$\displaystyle\therefore-\frac{2^{l+1}\pi^{l+1}}{n}\leqq\frac{l+1}{n}\sum_{k=0}^{2n-1}(-1)^k\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx\leqq\frac{2^{l+1}\pi^{l+1}}{n}$

$\displaystyle\lim_{n\to\infty}\frac{2^{l+1}\pi^{l+1}}{n}=0$ であるから、はさみうちの原理により　$\displaystyle\therefore\lim_{n\to\infty}\frac{l+1}{n}\sum_{k=0}^{2n-1}(-1)^k\int_{\frac{\pi k}{n}}^{\frac{\pi(k+1)}{n}}x^l\cos nxdx=0$

また、$\displaystyle\frac{2}{n}\sum_{k=0}^{2n-1}\left(\frac{k}{n}\right)^{l+1}<\frac{1}{n}\sum_{k=0}^{2n-1}\left\{\left(\frac{k+1}{n}\right)^{l+1}+\left(\frac{k}{n}\right)^{l+1}\right\}<\frac{2}{n}\sum_{k=0}^{2n-1}\left(\frac{k+1}{n}\right)^{l+1}$ であり、

$\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n-1}\left(\frac{k}{n}\right)^{l+1}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n-1}\left(\frac{k+1}{n}\right)^{l+1}=\int_0^2x^{l+1}dx=\left[\frac{x^{l+2}}{l+2}\right]_0^2=\frac{2^{l+2}}{l+2}$ であるから、

$$\therefore\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n-1}\left\{\left(\frac{k+1}{n}\right)^{l+1}+\left(\frac{k}{n}\right)^{l+1}\right\}=\frac{2^{l+3}}{l+2}\qquad\therefore\lim_{n\to\infty}W_n=2\pi^{l+2}\cdot\frac{2^{l+3}}{l+2}=\frac{2^{l+4}\pi^{l+2}}{l+2}\quad\cdots\cdots(答)$$