2008年東大文 1

$$\int_{-1}^{1} f(x)dx = 2\int_{0}^{1}(x^2+\alpha\beta)dx = 2\left[\frac{x^3}{3}+\alpha\beta x\right]_0^1 = \frac{2}{3}+2\alpha\beta = 1 \quad \therefore \alpha\beta = \frac{1}{6} \quad \text{——①}$$

$$S = \int_0^{\alpha} f(x)dx = \int_0^{\alpha}\left\{x^2-(\alpha+\beta)x+\alpha\beta\right\}dx = \left[\frac{x^3}{3}-(\alpha+\beta)\frac{x^2}{2}+\alpha\beta x\right]_0^{\alpha}$$

$$= \frac{1}{3}\alpha^3 - \frac{1}{2}(\alpha+\beta)\alpha^2 + \alpha^2\beta = \frac{1}{3}\alpha^3 - \frac{1}{2}\alpha^3 - \frac{1}{2}\alpha^2\beta + \alpha^2\beta = -\frac{1}{6}\alpha^3 + \frac{1}{2}\alpha^2\beta$$

①より　$\therefore S = -\frac{1}{6}\alpha^3 + \frac{1}{12}\alpha$ ……(答)

①より　$0 \leqq \alpha \leqq \beta = \frac{1}{6\alpha}$　$0 \leqq \alpha^2 \leqq \frac{1}{6}$　$\therefore 0 \leqq \alpha \leqq \frac{1}{\sqrt{6}}$ ——②

$f(\alpha) = -\frac{1}{6}\alpha^3 + \frac{1}{12}\alpha$ とすると　$f'(\alpha) = -\frac{1}{2}\alpha^2 + \frac{1}{12} = \frac{1}{2}\left(\frac{1}{6}-\alpha^2\right)$　②より　$\therefore f'(\alpha) \geqq 0$

②の範囲で S は単調増加であるから、S がとりうる値の最大値は

$$-\frac{1}{6}\cdot\frac{1}{6\sqrt{6}} + \frac{1}{12\sqrt{6}} = -\frac{\sqrt{6}}{216} + \frac{\sqrt{6}}{72} = \frac{3\sqrt{6}-\sqrt{6}}{216} = \frac{\sqrt{6}}{108}$$ ……(答)