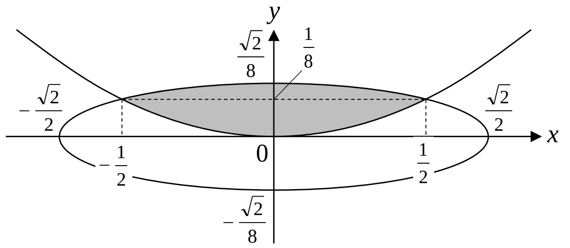
2012 年東大理 3

(1)

放物線 $y=\dfrac{1}{2}x^2$ と楕円 $\dfrac{x^2}{4}+4y^2=\dfrac{1}{8}$ の交点を求める。

$$\frac{x^2}{4}+4\cdot\frac{1}{4}x^4=\frac{1}{8}\qquad x^4+\frac{x^2}{4}-\frac{1}{8}=0\qquad \left(x^2+\frac{1}{2}\right)\left(x^2-\frac{1}{4}\right)=0$$

交点は $\left(\pm\dfrac{1}{2},\ \dfrac{1}{8}\right)$ であり、領域 S を図示すると右図の通り。



対称性により

$$V_1=2\pi\int_0^{\frac{1}{2}}\left(-\frac{x^2}{16}+\frac{1}{32}\right)dx-2\pi\int_0^{\frac{1}{2}}\frac{1}{4}x^4dx=2\pi\left[-\frac{x^3}{48}+\frac{1}{32}x\right]_0^{\frac{1}{2}}-2\pi\left[\frac{x^5}{20}\right]_0^{\frac{1}{2}}$$

$$=2\pi\left(-\frac{1}{384}+\frac{1}{64}-\frac{1}{640}\right)=2\pi\cdot\frac{-5+30-3}{1920}=\frac{11}{480}\pi\quad\cdots\cdots(\text{答})$$

$$V_2=\pi\int_{\frac{1}{8}}^{\frac{\sqrt{2}}{8}}\left(-16y^2+\frac{1}{2}\right)dy+\pi\int_0^{\frac{1}{8}}2ydy=\pi\left[-\frac{16}{3}y^3+\frac{1}{2}y\right]_{\frac{1}{8}}^{\frac{\sqrt{2}}{8}}+\pi\left[y^2\right]_0^{\frac{1}{8}}$$

$$=\pi\left(-\frac{\sqrt{2}}{48}+\frac{\sqrt{2}}{16}+\frac{1}{96}-\frac{1}{16}+\frac{1}{64}\right)=\left(\frac{\sqrt{2}}{24}-\frac{7}{192}\right)\pi=\frac{8\sqrt{2}-7}{192}\pi\quad\cdots\cdots(\text{答})$$

(2)

$$\frac{V_2}{V_1}=\frac{8\sqrt{2}-7}{192}\cdot\frac{480}{11}=\frac{5(8\sqrt{2}-7)}{22}<\frac{5(8\times1.42-7)}{22}=\frac{5\times4.36}{22}=\frac{21.8}{22}<1\qquad\therefore\frac{V_2}{V_1}<1\quad\cdots\cdots(\text{答})$$