2021 年東大理 3

(1)

$$f(x)=\frac{x}{x^2+3}\quad f'(x)=\frac{1\cdot(x^2+3)-x\cdot 2x}{(x^2+3)^2}=\frac{3-x^2}{(x^2+3)^2}$$

$f(1)=\dfrac{1}{4}\quad f'(1)=\dfrac{1}{8}\quad l$ の方程式は　$y=\dfrac{1}{8}(x-1)+\dfrac{1}{4}=\dfrac{1}{8}(x+1)$

$\dfrac{x}{x^2+3}=\dfrac{1}{8}(x+1)$ とすると　$8x=(x+1)(x^2+3)\quad x^3+x^2-5x+3=0\quad (x-1)^2(x+3)=0$

したがって、C と l の共有点で A と異なるものはただ 1 つであり、その座標は　$\left(-3,-\dfrac{1}{4}\right)$ ……(答)

(2)

$$\int_{\alpha}^{1}\{f(x)-g(x)\}^2dx=\int_{-3}^{1}\{f(x)\}^2dx-2\int_{-3}^{1}f(x)g(x)dx+\int_{-3}^{1}\{g(x)\}^2dx$$

$x=\sqrt{3}\tan\theta$ とおくと　$dx=\dfrac{\sqrt{3}d\theta}{\cos^2\theta}$

x	$-3\to 1$
θ	$-\dfrac{\pi}{3}\to\dfrac{\pi}{6}$

$$\int_{-3}^{1}\{f(x)\}^2dx=\int_{-3}^{1}\frac{x^2}{(x^2+3)^2}dx=\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}3\tan^2\theta\cdot\frac{1}{9}\cos^4\theta\cdot\frac{\sqrt{3}d\theta}{\cos^2\theta}=\frac{\sqrt{3}}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}\sin^2\theta\,d\theta=\frac{\sqrt{3}}{6}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}(1-\cos 2\theta)d\theta$$

$$=\frac{\sqrt{3}}{6}\left[\theta-\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}=\frac{\sqrt{3}}{6}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}+\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)=\frac{\sqrt{3}}{12}\pi-\frac{1}{4}$$

同様に

$$2\int_{-3}^{1}f(x)g(x)dx=\frac{1}{4}\int_{-3}^{1}\frac{x^2+x}{x^2+3}dx=\frac{1}{4}\int_{-3}^{1}\left(1+\frac{x}{x^2+3}-\frac{3}{x^2+3}\right)dx=\frac{1}{4}\int_{-3}^{1}\left(1+\frac{x}{x^2+3}\right)dx-\frac{1}{4}\int_{-3}^{1}\frac{3}{x^2+3}dx$$

$$=\frac{1}{4}\left[x+\frac{1}{2}\log(x^2+3)\right]_{-3}^{1}-\frac{1}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}\cos^2\theta\cdot\frac{\sqrt{3}d\theta}{\cos^2\theta}=\frac{1}{4}\left(1+\frac{1}{2}\log 4+3-\frac{1}{2}\log 12\right)-\frac{\sqrt{3}}{4}\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}}d\theta$$

$$=\frac{1}{4}\left(4-\frac{1}{2}\log 3\right)-\frac{\sqrt{3}}{4}\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=1-\frac{1}{8}\log 3-\frac{\sqrt{3}}{8}\pi$$

$$\int_{-3}^{1}\{g(x)\}^2dx=\frac{1}{64}\int_{-3}^{1}(x+1)^2dx=\frac{1}{64}\left[\frac{(x+1)^3}{3}\right]_{-3}^{1}=\frac{1}{64}\left(\frac{8}{3}+\frac{8}{3}\right)=\frac{1}{12}$$

以上により、求める値は

$$\frac{\sqrt{3}}{12}\pi-\frac{1}{4}-\left(1-\frac{1}{8}\log 3-\frac{\sqrt{3}}{8}\pi\right)+\frac{1}{12}=\frac{5\sqrt{3}}{24}\pi+\frac{1}{8}\log 3-\frac{7}{6}\ \cdots\cdots(\text{答})$$