2022年東大理 1

(1)

$$f'(x)=(-\sin x)\log(\cos x)+(\cos x)\cdot\frac{-\sin x}{\cos x}+\sin x+(\cos x)\log(\cos x)=(\cos x-\sin x)\log(\cos x)$$

$0\leqq x<\dfrac{\pi}{2}$ のとき、$0<\cos x\leqq 1$ であるから　$\log(\cos x)\leqq 0$

$\cos x-\sin x=\sqrt{2}\cos\left(x+\dfrac{\pi}{4}\right)$、$\dfrac{\pi}{4}\leqq x+\dfrac{\pi}{4}<\dfrac{3}{4}\pi$ であるから、$f(x)$ の増減は右の通りで、$x=\dfrac{\pi}{4}$ において最小値を持つ。(証明終)

x	0	$\cdots$	$\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{2}$
$f'(x)$		$-$	0	$+$	
$f(x)$		↘		↗	

(2)

$$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\log\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}+\int_0^{\frac{\pi}{4}}(\cos t)\log(\cos t)\,dt=-\frac{1}{2\sqrt{2}}\log 2-\frac{1}{\sqrt{2}}+\int_0^{\frac{\pi}{4}}(\cos t)\log(\cos t)\,dt$$

$I=\displaystyle\int_0^{\frac{\pi}{4}}(\cos t)\log(\cos t)\,dt$ を求める。$s=\sin t$ とおくと　$ds=\cos t\,dt$

t	$0\to\frac{\pi}{4}$
s	$0\to\frac{1}{\sqrt{2}}$

$I=\displaystyle\int_0^{\frac{1}{\sqrt{2}}}\log\sqrt{1-s^2}\,ds=\frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}}\log(1-s^2)\,ds$　ここで、不定積分 $\displaystyle\int\log(1-s^2)\,ds$ を求めておく。

$$\int\log(1-s^2)\,ds=s\log(1-s^2)-\int s\cdot\left(\frac{-2s}{1-s^2}\right)ds=s\log(1-s^2)+2\int\frac{s^2}{1-s^2}ds$$

$$=s\log(1-s^2)+2\int\left(\frac{1}{1-s^2}-1\right)ds=s\log(1-s^2)-2s+\int\left(\frac{1}{1+s}+\frac{1}{1-s}\right)ds$$

$$=s\log(1-s^2)-2s+\log(1+s)-\log(1-s)+C=s\log(1-s^2)-2s+\log\frac{1+s}{1-s}+\mathrm{C}$$

$$I=\frac{1}{2}\left[s\log(1-s^2)-2s+\log\frac{1+s}{1-s}\right]_0^{\frac{1}{\sqrt{2}}}=\frac{1}{2\sqrt{2}}\log\frac{1}{2}-\frac{1}{\sqrt{2}}+\frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}=-\frac{1}{2\sqrt{2}}\log 2-\frac{1}{\sqrt{2}}+\frac{1}{2}\log\left(\sqrt{2}+1\right)^2$$

$$=-\frac{1}{2\sqrt{2}}\log 2-\frac{1}{\sqrt{2}}+\log(\sqrt{2}+1)$$

求める最小値は　$-\dfrac{1}{2\sqrt{2}}\log 2-\dfrac{1}{\sqrt{2}}-\dfrac{1}{2\sqrt{2}}\log 2-\dfrac{1}{\sqrt{2}}+\log\left(\sqrt{2}+1\right)=-\dfrac{1}{\sqrt{2}}\log 2-\sqrt{2}+\log\left(\sqrt{2}+1\right)$　⋯⋯(答)