2024 年東大理 2

(1)

$$f(x)=\int_0^x \frac{x-t}{1+t^2}dt+\int_x^1 \frac{t-x}{1+t^2}dt=x\int_0^x \frac{1}{1+t^2}dt-\int_0^x \frac{t}{1+t^2}dt+\int_x^1 \frac{t}{1+t^2}dt-x\int_x^1 \frac{1}{1+t^2}dt$$

$$f'(x)=\int_0^x \frac{1}{1+t^2}dt+\frac{x}{1+x^2}-\frac{x}{1+x^2}-\frac{x}{1+x^2}-\int_x^1 \frac{1}{1+t^2}dt+\frac{x}{1+x^2}=\int_0^x \frac{1}{1+t^2}dt-\int_x^1 \frac{1}{1+t^2}dt$$

$t=\tan\theta$ とすると　$dt=\dfrac{d\theta}{\cos^2\theta}$　さらに$x=\tan\alpha$ とすると

$$f'(x)=\int_0^\alpha \frac{1}{1+\tan^2\theta}\cdot\frac{d\theta}{\cos^2\theta}-\int_\alpha^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta}\cdot\frac{d\theta}{\cos^2\theta}=\int_0^\alpha d\theta-\int_\alpha^{\frac{\pi}{4}} d\theta=[\theta]_0^\alpha-[\theta]_\alpha^{\frac{\pi}{4}}=2\alpha-\frac{\pi}{4}$$

したがって、$f'(\tan\alpha)=0$ のとき　$2\alpha-\dfrac{\pi}{4}=0$　$\therefore \alpha=\dfrac{\pi}{8}$ ……(答)

(2)

倍角公式により　$\tan 2\alpha=\tan\dfrac{\pi}{4}=1=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$　$1-\tan^2\alpha=2\tan\alpha$　$\tan^2\alpha+2\tan\alpha-1=0$

$t^2+2t-1=0$ を解くと　$t=-1\pm\sqrt{2}$　$0<\tan\alpha<1$ より　$\therefore \tan\alpha=-1+\sqrt{2}$ ……(答)

(3)

$f'(x)=\displaystyle\int_0^x \frac{1}{1+t^2}dt-\int_x^1 \frac{1}{1+t^2}dt$ であり、$\displaystyle\int_0^x \frac{1}{1+t^2}dt, -\int_x^1 \frac{1}{1+t^2}dt$ はそれぞれ単調増加であるから、

$f'(x)$は単調増加である。$f(x)$の増減は右の通り。

x	0	…	$-1+\sqrt{2}$	…	1
$f'(x)$		$-$	0	$+$	
$f(x)$		↘		↗	

$f(x)=\displaystyle\int_x^1 \frac{t}{1+t^2}dt-\int_0^x \frac{t}{1+t^2}dt+xf'(x)$より、

最小値は

$$f(\tan\alpha)=\int_{-1+\sqrt{2}}^1 \frac{t}{1+t^2}dt-\int_0^{-1+\sqrt{2}} \frac{t}{1+t^2}dt$$

$$=\left[\frac{1}{2}\log(1+t^2)\right]_{-1+\sqrt{2}}^1-\left[\frac{1}{2}\log(1+t^2)\right]_0^{-1+\sqrt{2}}=\frac{1}{2}\log 2-\frac{1}{2}\log\left(4-2\sqrt{2}\right)-\frac{1}{2}\log\left(4-2\sqrt{2}\right)$$

$$=\log\frac{\sqrt{2}}{4-2\sqrt{2}}=\log\frac{1}{2(\sqrt{2}-1)}=\log\frac{1+\sqrt{2}}{2}$$

$f(0)=\displaystyle\int_0^1 \frac{t}{1+t^2}dt, f(1)=\int_0^1 \frac{1}{1+t^2}dt-\int_0^1 \frac{t}{1+t^2}dt$　$\displaystyle\int_0^1 \frac{t}{1+t^2}dt=\left[\frac{1}{2}\log(1+t^2)\right]_0^1=\frac{1}{2}\log 2$

$t=\tan\theta$ とすると　$dt=\dfrac{d\theta}{\cos^2\theta}$　$\displaystyle\int_0^1 \frac{1}{1+t^2}dt=\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta}\cdot\frac{d\theta}{\cos^2\theta}=\int_0^{\frac{\pi}{4}} d\theta=[\theta]_0^{\frac{\pi}{4}}=\frac{\pi}{4}$

$\therefore f(0)=\dfrac{1}{2}\log 2, f(1)=\dfrac{\pi}{4}-\dfrac{1}{2}\log 2$　$0.345<f(0)<0.35, f(1)>\dfrac{3}{4}-0.35=0.4$ より　$\therefore f(1)>f(0)$

以上により、最大値は$\dfrac{\pi}{4}-\dfrac{1}{2}\log 2$、最小値は$\log\dfrac{1+\sqrt{2}}{2}$ ……(答)