2026 年東大理 $\boxed{1}$

(1)

$$f(\theta)=\sin\theta-\theta+\frac{\theta^3}{6}\quad f'(\theta)=\cos\theta-1+\frac{\theta^2}{2}\quad f''(\theta)=-\sin\theta+\theta\quad f'''(\theta)=-\cos\theta+1\geqq 0$$

$f''(\theta)$は$-1\leqq\theta\leqq 1$において単調増加。

$$f''(-1)=\sin 1-1<0\quad f''(0)=0\quad f''(1)=-\sin 1+1>0$$

$f'(\theta)$の増減は右の通りで、$\theta=0$において極小となる。
これより、$-1\leqq\theta\leqq 1$において$f'(\theta)\geqq 0$であり、
$f(\theta)$は単調増加であるから

θ	-1	$\cdots$	0	$\cdots$	1
$f''(\theta)$		$-$	0	$+$	
$f'(\theta)$		↘	0	↗	

$$\therefore M=f(1)=\sin 1-\frac{5}{6},m=f(-1)=-\sin 1+\frac{5}{6}\ \cdots\cdots(答)$$

(2)

$I=\displaystyle\int_0^{2\pi}\sin(\cos x-x)dx$とすると　$I=\displaystyle\int_0^{2\pi}\sin(\cos x)\cos x\,dx-\int_0^{2\pi}\cos(\cos x)\sin x\,dx$

$\displaystyle\int_0^{2\pi}\cos(\cos x)\sin x\,dx=[-\sin(\cos x)]_0^{2\pi}=0$ より　$\displaystyle\therefore I=\int_0^{2\pi}\sin(\cos x)\cos x\,dx$

(1)より、$0\leqq x\leqq 2\pi$において$-1\leqq\cos x\leqq 1$、$-M\leqq f(\cos x)\leqq M$である。

$0\leqq x\leqq\dfrac{\pi}{2}$のとき　$0\leqq\cos x\leqq 1$、$0\leqq f(\cos x)\leqq M$であるから

$$0\leqq f(\cos x)\cos x\leqq M\cos x$$

$$\cos^2 x-\frac{\cos^4 x}{6}\leqq\sin(\cos x)\cos x\leqq\cos^2 x-\frac{\cos^4 x}{6}+M\cos x$$

$$\int_0^{\frac{\pi}{2}}\left(\cos^2 x-\frac{\cos^4 x}{6}\right)dx\leqq\int_0^{\frac{\pi}{2}}\sin(\cos x)\cos x\,dx\leqq\int_0^{\frac{\pi}{2}}\left(\cos^2 x-\frac{\cos^4 x}{6}+M\cos x\right)dx$$

不定積分$\displaystyle\int\left(\cos^2 x-\frac{\cos^4 x}{6}\right)dx$を求めておく。

$$\int\left(\cos^2 x-\frac{\cos^4 x}{6}\right)dx=\int\left\{\frac{1+\cos 2x}{2}-\frac{(1+\cos 2x)^2}{24}\right\}dx=\int\left(\frac{11}{24}+\frac{5}{12}\cos 2x-\frac{1+\cos 4x}{48}\right)dx$$

$$=\int\left(\frac{7}{16}+\frac{5}{12}\cos 2x-\frac{1}{48}\cos 4x\right)dx=\frac{7}{16}x+\frac{5}{24}\sin 2x-\frac{1}{192}\cos 4x+C$$

$$\left[\frac{7}{16}x+\frac{5}{24}\sin 2x-\frac{1}{192}\cos 4x\right]_0^{\frac{\pi}{2}}\leqq\int_0^{\frac{\pi}{2}}\sin(\cos x)\cos x\,dx\leqq\left[\frac{7}{16}x+\frac{5}{24}\sin 2x-\frac{1}{192}\cos 4x+M\sin x\right]_0^{\frac{\pi}{2}}$$

$$\therefore\frac{7}{32}\pi\leqq\int_0^{\frac{\pi}{2}}\sin(\cos x)\cos x\,dx\leqq\frac{7}{32}\pi+M\text{——①}$$

$\dfrac{\pi}{2} \leqq x \leqq \dfrac{3\pi}{2}$ のとき $-1 \leqq \cos x \leqq 0$、$-M \leqq f(\cos x) \leqq 0$ であるから

$$0 \leqq f(\cos x)\cos x \leqq -M\cos x$$

$$\cos^2 x - \frac{\cos^4 x}{6} \leqq \sin(\cos x)\cos x \leqq \cos^2 x - \frac{\cos^4 x}{6} - M\cos x$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\cos^2 x - \frac{\cos^4 x}{6}\right) dx \leqq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\cos x)\cos x\, dx \leqq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\cos^2 x - \frac{\cos^4 x}{6} - M\cos x\right) dx$$

$$\left[\frac{7}{16}x + \frac{5}{24}\sin 2x - \frac{1}{192}\cos 4x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \leqq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\cos x)\cos x\, dx \leqq \left[\frac{7}{16}x + \frac{5}{24}\sin 2x - \frac{1}{192}\cos 4x - M\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$\therefore \frac{7}{16}\pi \leqq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\cos x)\cos x\, dx \leqq \frac{7}{16}\pi + 2M \text{——②}$$

$\dfrac{3\pi}{2} \leqq x \leqq 2\pi$ のとき　$0 \leqq \cos x \leqq 1$、$0 \leqq f(\cos x) \leqq M$ であるから

$$0 \leqq f(\cos x)\cos x \leqq M\cos x$$

$$\cos^2 x - \frac{\cos^4 x}{6} \leqq \sin(\cos x)\cos x \leqq \cos^2 x - \frac{\cos^4 x}{6} + M\cos x$$

$$\int_{\frac{3\pi}{2}}^{2\pi} \left(\cos^2 x - \frac{\cos^4 x}{6}\right) dx \leqq \int_{\frac{3\pi}{2}}^{2\pi} \sin(\cos x)\cos x\, dx \leqq \int_{\frac{3\pi}{2}}^{2\pi} \left(\cos^2 x - \frac{\cos^4 x}{6} + M\cos x\right) dx$$

$$\left[\frac{7}{16}x + \frac{5}{24}\sin 2x - \frac{1}{192}\cos 4x\right]_{\frac{3\pi}{2}}^{2\pi} \leqq \int_{\frac{3\pi}{2}}^{2\pi} \sin(\cos x)\cos x\, dx \leqq \left[\frac{7}{16}x + \frac{5}{24}\sin 2x - \frac{1}{192}\cos 4x + M\sin x\right]_{\frac{3\pi}{2}}^{2\pi}$$

$$\therefore \frac{7}{32}\pi \leqq \int_{\frac{3\pi}{2}}^{2\pi} \sin(\cos x)\cos x\, dx \leqq \frac{7}{32}\pi + M \text{——③}$$

①～③を辺々足すと　$\dfrac{7}{8}\pi \leqq \displaystyle\int_0^{2\pi} \sin(\cos x)\cos x\, dx \leqq \dfrac{7}{8}\pi + 4M$　$\therefore \dfrac{7}{8}\pi \leqq I \leqq \dfrac{7}{8}\pi + 4M$　(証明終)