1章　ベクトル解析

§2　スカラー場とベクトル場　(p.9〜p.)

BASIC

30（1）

$$\frac{\partial \varphi}{\partial x} = 2xz$$

$$\frac{\partial \varphi}{\partial y} = 3y^2 z$$

$$\frac{\partial \varphi}{\partial z} = x^2 + y^3$$

よって，$\nabla\varphi = (2zx,\ 3y^2 z,\ x^2 + y^3)$

したがって

$$(\nabla\varphi)_{\mathrm{P}} = (2\cdot 2\cdot 1,\ \ 3\cdot(-1)^2\cdot 2,\ \ 1^2+(-1)^3)$$

$$= \mathbf{(4,\ 6,\ 0)}$$

（2）

$$|(\nabla\varphi)_{\mathrm{P}}| = \sqrt{4^2+6^2+0}$$

$$= \sqrt{52} = 2\sqrt{13}$$

よって

$$\boldsymbol{n} = \frac{1}{2\sqrt{13}}(4,\ 6,\ 0)$$

$$= \boldsymbol{\frac{1}{\sqrt{13}}(2,\ 3,\ 0)}$$

（3）

$$(\nabla\varphi)_{\mathrm{P}}\cdot\boldsymbol{n} = \frac{1}{\sqrt{13}}(4\cdot 2 + 6\cdot 3 + 0)$$

$$= \frac{1}{\sqrt{13}}\cdot 26$$

$$= \frac{26\sqrt{13}}{13} = \boldsymbol{2\sqrt{13}}$$

（4）　$|\boldsymbol{a}| = \sqrt{3^2+0+(-4)^2} = \sqrt{25} = 5$ であるから，$\boldsymbol{a}$ と同じ向きの単位ベクトル $\boldsymbol{e}$ とすると

$$\boldsymbol{e} = \frac{1}{5}(3,\ 0,\ -4)$$

よって，求める方向微分係数は

$$(\nabla\varphi)_{\mathrm{P}}\cdot\boldsymbol{e} = \frac{1}{5}(4\cdot 3+0+0)$$

$$= \frac{1}{5}\cdot 12 = \boldsymbol{\frac{12}{5}}$$

31（1）

$$\text{左辺} = \nabla(a\varphi) + \nabla(b\psi)$$

$$= a\nabla\varphi + b\nabla\psi = \text{右辺}$$

（2）

$$\text{左辺} = (\nabla\varphi)\varphi + \varphi(\nabla\varphi)$$

$$= \varphi\nabla\varphi + \varphi\nabla\varphi$$

$$= 2\varphi\nabla\varphi = \text{右辺}$$

32（1）

$$\frac{\partial \varphi}{\partial x} = -\frac{yz}{(xyz)^2} = -\frac{yz}{x^2y^2z^2}$$

$$\frac{\partial \varphi}{\partial y} = -\frac{zx}{(xyz)^2} = -\frac{zx}{x^2y^2z^2}$$

$$\frac{\partial \varphi}{\partial z} = -\frac{xy}{(xyz)^2} = -\frac{xy}{x^2y^2z^2}$$

よって，$\boldsymbol{\nabla\varphi = -\frac{1}{x^2y^2z^2}(yz,\ zx,\ xy)}$

（2）

$$\frac{\partial \varphi}{\partial x} = \frac{1(x+y)-(x+z)\cdot 1}{(x+y)^2} = \frac{y-z}{(x+y)^2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{0(x+y)-(x+z)\cdot 1}{(x+y)^2} = \frac{-x-z}{(x+y)^2}$$

$$\frac{\partial \varphi}{\partial z} = \frac{1(x+y)-(x+z)\cdot 0}{(x+y)^2} = \frac{x+y}{(x+y)^2}$$

よって，$\boldsymbol{\nabla\varphi = \frac{1}{(x+y)^2}(y-z,\ -z-x,\ x+y)}$

33（1）

$$\nabla\cdot\boldsymbol{a} = \frac{\partial}{\partial x}(zx) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz)$$

$$= z + x + y = \boldsymbol{x+y+z}$$

$$\nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ zx & xy & yz \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(yz)\,\boldsymbol{i} + \frac{\partial}{\partial z}(zx)\,\boldsymbol{j} + \frac{\partial}{\partial x}(xy)\,\boldsymbol{k}$$

$$-\left\{\frac{\partial}{\partial z}(xy)\,\boldsymbol{i} + \frac{\partial}{\partial x}(yz)\,\boldsymbol{j} + \frac{\partial}{\partial y}(zx)\,\boldsymbol{k}\right\}$$

$$= z\,\boldsymbol{i} + x\,\boldsymbol{j} + y\,\boldsymbol{k}$$

$$= \boldsymbol{(z,\ x,\ y)}$$

（2）

$$\nabla\cdot\boldsymbol{a} = \frac{\partial}{\partial x}(z^2y) + \frac{\partial}{\partial y}(-z^2x) + \frac{\partial}{\partial z}(x+y)$$

$$= 0+0+0 = \mathbf{0}$$

$$\nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2y & -z^2x & x+y \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(x+y)\,\boldsymbol{i} + \frac{\partial}{\partial z}(z^2y)\,\boldsymbol{j} + \frac{\partial}{\partial x}(-z^2x)\,\boldsymbol{k}$$

$$-\left\{\frac{\partial}{\partial z}(-z^2x)\,\boldsymbol{i} + \frac{\partial}{\partial x}(x+y)\,\boldsymbol{j} + \frac{\partial}{\partial y}(z^2y)\,\boldsymbol{k}\right\}$$

$$= 1\,\boldsymbol{i} + 2zy\,\boldsymbol{j} - z^2\,\boldsymbol{k} - \{(-2zx)\,\boldsymbol{i} + 1\,\boldsymbol{j} + z^2\,\boldsymbol{k}\}$$

$$= (2zx+1)\,\boldsymbol{i} + (2zy-1)\,\boldsymbol{j} - 2z^2\,\boldsymbol{k}$$

$$= \boldsymbol{(2zx+1,\ 2yz-1,\ -2z^2)}$$

34　$\boldsymbol{a} = e^{xy}(x,\ y,\ z^2)$ であるから

$$\nabla\cdot\boldsymbol{a} = \nabla(e^{xy})\cdot(x,\ y,\ z^2) + e^{xy}\{\nabla\cdot(x,\ y,\ z^2)\}$$

$$= (ye^{xy},\ xe^{xy},\ 0)\cdot(x,\ y,\ z^2) + e^{xy}(1+1+2z)$$

$$= xye^{xy} + xye^{xy} + 0 + (2+2z)e^{xy}$$

$$= 2xye^{xy} + (2+2z)e^{xy}$$

$$= \boldsymbol{2e^{xy}(xy+z+1)}$$

$$\nabla\times\boldsymbol{a} = \nabla(e^{xy})\times(x,\ y,\ z^2) + e^{xy}\{\nabla\times(x,\ y,\ z^2)\}$$

$$= (ye^{xy},\ xe^{xy},\ 0)\times(x,\ y,\ z^2)$$

$$+\,e^{xy}\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ ye^{xy} & xe^{xy} & 0 \\ x & y & z^2 \end{vmatrix} + 0\,\boldsymbol{i} + 0\,\boldsymbol{j} + 0\,\boldsymbol{k}$$

$$= xz^2e^{xy}\,\boldsymbol{i} + 0\,\boldsymbol{j} + y^2e^{xy}\,\boldsymbol{k}$$

$$-\{0\,\boldsymbol{i} + yz^2e^{xy}\,\boldsymbol{j} + x^2e^{xy}\,\boldsymbol{k}\}$$

$$= xz^2e^{xy}\,\boldsymbol{i} - yz^2e^{xy}\,\boldsymbol{j} + (y^2e^{xy} - x^2e^{xy})\,\boldsymbol{k}$$

$$= \boldsymbol{e^{xy}(xz^2,\ -yz^2,\ y^2-x^2)}$$

35　$\nabla\varphi = (yz^2,\ xz^2,\ 2xyz)$ であるから，

$$\varphi\nabla\varphi = xyz^2(yz^2,\ xz^2,\ 2xyz)$$

$$= (xy^2z^4,\ x^2yz^4,\ 2x^2y^2z^3)$$

よって

$$\nabla(\varphi\nabla\varphi) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z^4 & x^2yz^4 & 2x^2y^2z^3 \end{vmatrix}$$

$$= 4x^2yz^3\,\boldsymbol{i} + 4xy^2z^3\,\boldsymbol{j} + 2xyz^4\,\boldsymbol{k}$$

$$\quad - (4x^2yz^3\,\boldsymbol{i} + 4xy^2z^3\,\boldsymbol{j} + 2xyz^4\,\boldsymbol{k})$$

$$= (0,\ 0,\ 0) = \boldsymbol{0}$$

36（1）$\dfrac{1}{r^2} = \dfrac{1}{|\,\boldsymbol{r}\,|^2} = \dfrac{1}{x^2+y^2+r^2}$ であるから

$$\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) = -\frac{2x}{(x^2+y^2+z^2)^2} = -\frac{2x}{r^4}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r^2}\right) = -\frac{2y}{(x^2+y^2+z^2)^2} = -\frac{2y}{r^4}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r^2}\right) = -\frac{2z}{(x^2+y^2+z^2)^2} = -\frac{2z}{r^4}$$

よって

$$与式 = \left(-\frac{2x}{r^4},\ -\frac{2y}{r^4},\ -\frac{2z}{r^4}\right)$$

$$= -\frac{2}{r^4}(x,\ y,\ z) = -\frac{2\boldsymbol{r}}{r^4}$$

〔別解〕

$r = \sqrt{x^2+y^2+r^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$ であるから

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2y = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

よって，$\nabla r = \left(\dfrac{x}{r},\ \dfrac{y}{r},\ \dfrac{z}{r}\right) = \dfrac{1}{r}(x,\ y,\ z) = \dfrac{\boldsymbol{r}}{r}$

したがって

$$与式 = \nabla(r^{-2})$$

$$= -2r^{-3}(\nabla r)$$

$$= -\frac{2}{r^3}\cdot\frac{\boldsymbol{r}}{r} = -\frac{2\boldsymbol{r}}{r^4}$$

（2）$\log r = \log(\sqrt{x^2+y^2+z^2})$ であるから

$$\frac{\partial}{\partial x}(\log r) = \frac{\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{x^2+y^2+z^2} = \frac{x}{r^2}$$

$$\frac{\partial}{\partial y}(\log r) = \frac{\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{x^2+y^2+z^2} = \frac{y}{r^2}$$

$$\frac{\partial}{\partial z}(\log r) = \frac{\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{x^2+y^2+z^2} = \frac{z}{r^2}$$

よって

$$与式 = \left(\frac{x}{r^2},\ \frac{y}{r^2},\ \frac{z}{r^2}\right)$$

$$= \frac{1}{r^2}(x,\ y,\ z) = \frac{\boldsymbol{r}}{r^2}$$

〔別解〕

$r = \sqrt{x^2+y^2+r^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$ であるから

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2y = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot 2z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

よって，$\nabla r = \left(\dfrac{x}{r},\ \dfrac{y}{r},\ \dfrac{z}{r}\right) = \dfrac{1}{r}(x,\ y,\ z) = \dfrac{\boldsymbol{r}}{r}$

したがって

$$与式 = \frac{1}{r}(\nabla r)$$

$$= \frac{1}{r}\cdot\frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{r}}{r^2}$$

37（1）$\dfrac{\partial r}{\partial x} = 2x$ より，$\dfrac{\partial^2 r}{\partial x^2} = 2$

$\dfrac{\partial r}{\partial y} = 2y$ より，$\dfrac{\partial^2 r}{\partial y^2} = 2$

$\dfrac{\partial r}{\partial z} = 2z$ より，$\dfrac{\partial^2 r}{\partial z^2} = 2$

よって

$$\nabla^2\varphi = 2+2+2 = \boldsymbol{6}$$

（2）$\dfrac{\partial r}{\partial x} = 2xyz + y^2z + yz^2$ より，$\dfrac{\partial^2 r}{\partial x^2} = 2yz$

$\dfrac{\partial r}{\partial y} = x^2z + 2xyz + xz^2$ より，$\dfrac{\partial^2 r}{\partial y^2} = 2xz$

$\dfrac{\partial r}{\partial z} = x^2y + xy^2 + 2xyz$ より，$\dfrac{\partial^2 r}{\partial z^2} = 2xy$

よって

$$\nabla^2\varphi = \boldsymbol{2yz + 2zx + 2xy}$$

（3）$\dfrac{\partial r}{\partial x} = 3x^2y - yz^2$ より，$\dfrac{\partial^2 r}{\partial x^2} = 6xy$

$\dfrac{\partial r}{\partial y} = x^3 - xz^2$ より，$\dfrac{\partial^2 r}{\partial y^2} = 0$

$\dfrac{\partial r}{\partial z} = -2xyz$ より，$\dfrac{\partial^2 r}{\partial z^2} = -2xy$

よって

$$\nabla^2\varphi = 6xy - 2xy = \boldsymbol{4xy}$$

（4）$\dfrac{\partial r}{\partial x} = (x)'ye^z\log x + xye^z(\log x)'$

$$= ye^z\log x + xye^z\cdot\frac{1}{x}$$

$= ye^z\log x + ye^z$　より

$$\frac{\partial^2 r}{\partial x^2} = ye^z\cdot\frac{1}{x} = \frac{y}{x}e^z$$

$\dfrac{\partial r}{\partial y} = xe^z\log x$ より，$\dfrac{\partial^2 r}{\partial y^2} = 0$

$\dfrac{\partial r}{\partial z} = xye^z\log x$ より，$\dfrac{\partial^2 r}{\partial z^2} = xye^z\log x$

よって

$$\nabla^2\varphi = \frac{y}{x}e^z + 0 + xye^z\log x$$

$$= \left(\boldsymbol{\frac{y}{x} + xy\log x}\right)\boldsymbol{e^z}$$

■