2章　偏微分

BASIC

43（1）　$z=2+3x-y$ より，$3x-y-z+2=0$

よって，法線ベクトルの 1 つは，$\boldsymbol{(3,\ -1,\ -1)}$

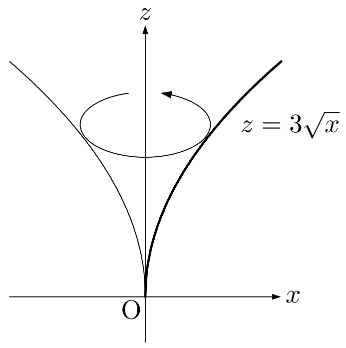
（2）　$2x+3y+z=1$ より，$2x+3y+z-1=0$

よって，法線ベクトルの 1 つは，$\boldsymbol{(2,\ 3,\ 1)}$

44　　立体的な図は，解答を参考にしてください．

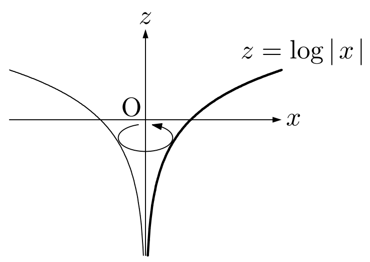
（1）　$y=0$, $(x \geqq 0)$ とすれば，$z=3(x^2)^{\frac{1}{4}}=3x^{\frac{1}{2}}=3\sqrt{x}$

よって，求める曲面は，zx 平面上のこの曲線を，z 軸のまわりに回転してできる回転面である．



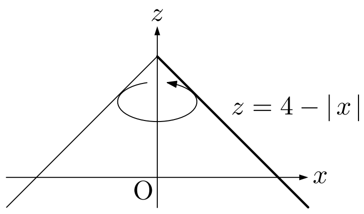
（2）　$y=0$ とすれば，$z=\log\sqrt{x^2}=\log|x|$

よって，求める曲面は，zx 平面上のこの曲線を，z 軸のまわりに回転してできる回転面である．



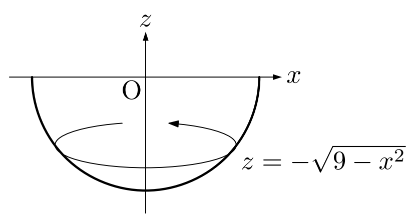
（3）　$y=0$ とすれば，$z=4-\sqrt{x^2}=4-|x|$

よって，求める曲面は，zx 平面上のこの曲線を，z 軸のまわりに回転してできる回転面である．



（4）　$y=0$ とすれば，$z=-\sqrt{9-x^2}$　$(-3 \leqq x \leqq 3)$

これより，$x^2+z^2=3^2$，$z \leqq 0$ であるから，求める曲面は，図のような半円を，z 軸のまわりに回転してできる回転面である．



45（1）$z_x=4\cdot 2x-3y$

$=\boldsymbol{8x-3y}$

$z_y=-3x+6\cdot 2y$

$=\boldsymbol{-3x+12y}$

（2）$z_x=5y\cdot 2x+3y^2$

$=\boldsymbol{10xy+3y^2}$

$z_y=5x^2+3x\cdot 3y^2$

$=\boldsymbol{5x^2+9xy^2}$

（3）$z_x=\dfrac{1}{2}(2x^2y+3xy^2)^{-\frac{1}{2}}\cdot(2y\cdot 2x+3y^2)$

$=\boldsymbol{\dfrac{4xy+3y^2}{2\sqrt{2x^2y+3xy^2}}}$

$z_y=\dfrac{1}{2}(2x^2y+3xy^2)^{-\frac{1}{2}}\cdot(2x^2+3x\cdot 2y)$

$=\dfrac{2x^2+6xy}{2\sqrt{2x^2y+3xy^2}}$

$=\boldsymbol{\dfrac{x^2+3xy}{\sqrt{2x^2y+3xy^2}}}$

（4）$z_x=e^{xy}\cdot y$

$=\boldsymbol{ye^{xy}}$

$z_y=e^{xy}\cdot x$

$=\boldsymbol{xe^{xy}}$

（5）$z_x=e^{3x}\cdot 3\cdot\tan 2y$

$=\boldsymbol{3e^{3x}\tan 2y}$

$z_y=e^{3x}\cdot\dfrac{1}{\cos^2 2y}\cdot 2$

$=\boldsymbol{\dfrac{2e^{3x}}{\cos^2 2y}}$

（6）$z_x=\cos 2x\cdot 2\cdot\log 3y$

$=\boldsymbol{2\cos 2x\log 3y}$

$z_y=\sin 2x\cdot\dfrac{1}{3y}\cdot 3$

$=\boldsymbol{\dfrac{\sin 2x}{y}}$

（7）$z_x=e^{2x+y}\cdot 2\cdot\cos(x-y)+e^{2x+y}\cdot\{-\sin(x-y)\cdot 1\}$

$=2e^{2x+y}\cos(x-y)-e^{2x+y}\sin(x-y)$

$=\boldsymbol{e^{2x+y}\{2\cos(x-y)-\sin(x-y)\}}$

$z_y=e^{2x+y}\cdot 1\cdot\cos(x-y)+e^{2x+y}\cdot\{-\sin(x-y)\cdot(-1)\}$

$=e^{2x+y}\cos(x-y)+e^{2x+y}\sin(x-y)$

$=\boldsymbol{e^{2x+y}\{\cos(x-y)+\sin(x-y)\}}$

（8）$z_x=1\cdot\log(2x+5y)+(x+3y)\cdot\dfrac{1}{2x+5y}\cdot 2$

$=2e^{2x+y}\cos(x-y)-e^{2x+y}\sin(x-y)$

$=\boldsymbol{\log(2x+5y)+\dfrac{2(x+3y)}{2x+5y}}$

$z_y=3\cdot\log(2x+5y)+(x+3y)\cdot\dfrac{1}{2x+5y}\cdot 5$

$=\boldsymbol{3\log(2x+5y)+\dfrac{5(x+3y)}{2x+5y}}$

（9）$z_x=\dfrac{1(3x-2y)-(x+2y)\cdot 3}{(3x-2y)^2}$

$=\dfrac{3x-2y-3x-6y}{(3x-2y)^2}$

$=\boldsymbol{\dfrac{-8y}{(3x-2y)^2}}$

$$z_y = \frac{2(3x-2y)-(x+2y)\cdot(-2)}{(3x-2y)^2}$$
$$= \frac{6x-4y+2x+4y}{(3x-2y)^2}$$
$$= \mathbf{\frac{8x}{(3x-2y)^2}}$$

(10) $$z_x = \frac{\cos x(\sin x+\cos y)-(\sin x-\cos y)\cdot\cos x}{(\sin x+\cos y)^2}$$
$$= \mathbf{\frac{2\cos x\cos y}{(\sin x+\cos y)^2}}$$
$$z_y = \frac{\sin y(\sin x+\cos y)-(\sin x-\cos y)\cdot(-\sin y)}{(\sin x+\cos y)^2}$$
$$= \mathbf{\frac{2\sin x\sin y}{(\sin x+\cos y)^2}}$$

46（1） $f_x(x,\ y) = 4x-y$

$f_y(x,\ y) = -x+6y$

これより

$f_x(1,\ 2) = 4\cdot 1-2 = \mathbf{2}$

$f_y(1,\ 2) = -1+6\cdot 2 = \mathbf{11}$

（2） $f_x(x,\ y) = e^{x^2y}\cdot 2xy = 2xye^{x^2y}$

$f_y(x,\ y) = e^{x^2y}\cdot x^2 = x^2e^{x^2y}$

これより

$f_x(1,\ 2) = 2\cdot 1\cdot 2\cdot e^{1^2\cdot 2} = \mathbf{4e^2}$

$f_y(1,\ 2) = 1^2\cdot e^{1^2\cdot 2} = \mathbf{e^2}$

（3） $f_x(x,\ y) = \dfrac{1}{x+y^2}\cdot 1 = \dfrac{1}{x+y^2}$

$f_y(x,\ y) = \dfrac{1}{x+y^2}\cdot 2y = \dfrac{2y}{x+y^2}$

これより

$f_x(1,\ 2) = \dfrac{1}{1+2^2} = \mathbf{\dfrac{1}{5}}$

$f_y(1,\ 2) = \dfrac{2\cdot 2}{1+2^2} = \mathbf{\dfrac{4}{5}}$

（4） $f_x(x,\ y) = \dfrac{1}{2}(xy^2+1)^{-\frac{1}{2}}\cdot y^2 = \dfrac{y^2}{2\sqrt{xy^2+1}}$

$f_y(x,\ y) = \dfrac{1}{2}(xy^2+1)^{-\frac{1}{2}}\cdot 2xy = \dfrac{xy}{\sqrt{xy^2+1}}$

これより

$f_x(1,\ 2) = \dfrac{2^2}{2\sqrt{1\cdot 2^2+1}} = \mathbf{\dfrac{2}{\sqrt{5}}}$

$f_y(1,\ 2) = \dfrac{1\cdot 2}{\sqrt{1\cdot 2^2+1}} = \mathbf{\dfrac{2}{\sqrt{5}}}$

47（1） $f_x(x,\ y,\ z) = \mathbf{2y+z}$

$f_y(x,\ y,\ z) = \mathbf{2x+3z}$

$f_z(x,\ y,\ z) = \mathbf{3y+x}$

これより

$f_x(1,\ 2,\ 1) = 2\cdot 2+1 = \mathbf{5}$

$f_y(1,\ 2,\ 1) = 2\cdot 1+3\cdot 1 = \mathbf{5}$

$f_z(1,\ 2,\ 1) = 3\cdot 2+1 = \mathbf{7}$

（2） $f_x(x,\ y,\ z) = 3(2x-3y+2z)^2\cdot 2$

$= \mathbf{6(2x-3y+2z)^2}$

$f_y(x,\ y,\ z) = 3(2x-3y+2z)^2\cdot(-3)$

$= \mathbf{-9(2x-3y+2z)^2}$

$f_z(x,\ y,\ z) = 3(2x-3y+2z)^2\cdot 2$

$= \mathbf{6(2x-3y+2z)^2}$

これより

$f_x(1,\ 2,\ 1) = 6(2\cdot 1-3\cdot 2+2\cdot 1)^2$

$= 6\cdot(-2)^2 = \mathbf{24}$

$f_y(1,\ 2,\ 1) = -9(2\cdot 1-3\cdot 2+2\cdot 1)^2$

$= -9\cdot(-2)^2 = \mathbf{-36}$

$f_z(1,\ 2,\ 1) = 6(2\cdot 1-3\cdot 2+2\cdot 1)^2$

$= 6\cdot(-2)^2 = \mathbf{24}$

（3） $f_x(x,\ y,\ z) = \mathbf{\dfrac{z}{y}}$

$f_y(x,\ y,\ z) = xz\cdot\left(-\dfrac{1}{y^2}\right) = \mathbf{-\dfrac{xz}{y^2}}$

$f_z(x,\ y,\ z) = \mathbf{\dfrac{x}{y}}$

これより

$f_x(1,\ 2,\ 1) = \mathbf{\dfrac{1}{2}}$

$f_y(1,\ 2,\ 1) = -\dfrac{1\cdot 1}{2^2} = \mathbf{-\dfrac{1}{4}}$

$f_z(1,\ 2,\ 1) = \mathbf{\dfrac{1}{2}}$

（4） $f_x(x,\ y,\ z) = e^{x^2+y^2+z^2}\cdot 2x = \mathbf{2xe^{x^2+y^2+z^2}}$

$f_y(x,\ y,\ z) = e^{x^2+y^2+z^2}\cdot 2y = \mathbf{2ye^{x^2+y^2+z^2}}$

$f_z(x,\ y,\ z) = e^{x^2+y^2+z^2}\cdot 2z = \mathbf{2ze^{x^2+y^2+z^2}}$

これより

$f_x(1,\ 2,\ 1) = 2\cdot 1\cdot e^{1^2+2^2+1^2} = \mathbf{2e^6}$

$f_y(1,\ 2,\ 1) = 2\cdot 2\cdot e^{1^2+2^2+1^2} = \mathbf{4e^6}$

$f_z(1,\ 2,\ 1) = 2\cdot 1\cdot e^{1^2+2^2+1^2} = \mathbf{2e^6}$

48（1） $z_x = 6x^2y^2-4y^3$

$z_y = 4x^3y-12xy^2$

よって

$dz = z_x dx + z_y dy$

$= \mathbf{(6x^2y^2-4y^3)dx+(4x^3y-12xy^2)dy}$

（2） $z_x = 4\sqrt{3y+2}$

$z_y = (4x+1)\cdot\dfrac{1}{2}(3y+2)^{-\frac{1}{2}}\cdot 3 = \dfrac{3(4x+1)}{2\sqrt{3y+2}}$

よって

$dz = z_x dx + z_y dy$

$= \mathbf{4\sqrt{3y+2}\,dx+\dfrac{3(4x+1)}{2\sqrt{3y+2}}\,dy}$

（3） $z_x = 4(3x+5y)^3\cdot 3 = 12(3x+5y)^3$

$z_y = 4(3x+5y)^3\cdot 5 = 20(3x+5y)^3$

よって

$dz = z_x dx + z_y dy$

$= \mathbf{12(3x+5y)^3\,dx+20(3x+5y)^3\,dy}$

（4） $z_x = \dfrac{1}{\cos^2(x^2+y^3)}\cdot 2x = \dfrac{2x}{\cos^2(x^2+y^3)}$

$z_y = \dfrac{1}{\cos^2(x^2+y^3)}\cdot 3y^2x = \dfrac{3y^2}{\cos^2(x^2+y^3)}$

よって

$dz = z_x dx + z_y dy$

$= \mathbf{\dfrac{2x}{\cos^2(x^2+y^3)}\,dx+\dfrac{3y^2}{\cos^2(x^2+y^3)}\,dy}$

（5） $z_x = 2e^{x+3y}+(2x+y)e^{x+3y}\cdot 1$

$= (2+2x+y)e^{x+3y}$

$z_y = 1 \cdot e^{x+3y} + (2x+y) \cdot e^{x+3y} \cdot 3$

$= (1+6x+3y)e^{x+3y}$

よって

$dz = z_x dx + z_y dy$

$= \boldsymbol{(2x+y+2)e^{x+3y}\,dx}$

$\boldsymbol{+(6x+3y+1)e^{x+3y}\,dy}$

（6）　$z_x = \dfrac{2(x^2+y^2) - (2x-3y) \cdot 2x}{(x^2+y^2)^2}$

$= \dfrac{2x^2+2y^2-4x^2+6xy}{(x^2+y^2)^2}$

$= \dfrac{-2x^2+6xy+2y^2}{(x^2+y^2)^2}$

$z_y = \dfrac{-3(x^2+y^2) - (2x-3y) \cdot 2y}{(x^2+y^2)^2}$

$= \dfrac{-3x^2-3y^2-4xy+6y^2}{(x^2+y^2)^2}$

$= \dfrac{-3x^2-4xy+3y^2}{(x^2+y^2)^2}$

よって

$dz = z_x dx + z_y dy$

$= \boldsymbol{\dfrac{-2x^2+6xy+2y^2}{(x^2+y^2)^2}\,dx}$

$\boldsymbol{+\dfrac{-3x^2-4xy+3y^2}{(x^2+y^2)^2}\,dy}$

49　題意より

$S = \pi x^2 \times 2 + y \times 2\pi x$

$= 2\pi x^2 + 2\pi xy$

これより

$\dfrac{\partial S}{\partial x} = 4\pi x + 2\pi y = 2\pi(2x+y)$

$\dfrac{\partial S}{\partial y} = 2\pi x$

よって，$\varDelta S \fallingdotseq \dfrac{\partial S}{\partial x}\varDelta x + \dfrac{\partial S}{\partial y}\varDelta y$

$= \boldsymbol{2\pi(2x+y)\,\varDelta x + 2\pi x\,\varDelta y}$

50（1）　$z_x = 2x,\ \ z_y = 4y$

これより，$x=1,\ y=1$ のとき，$z_x = 2,\ z_y = 4$ であるから，求める接平面の方程式は

$z - 3 = 2(x-1) + 4(y-1)$

整理して

$z - 3 = 2x - 2 + 4y - 4$

$\boldsymbol{2x + 4y - z = 3}$

（2）　$z_x = \dfrac{1}{2}(5-x^2y^2)^{-\frac{1}{2}} \cdot (-2xy^2) = \dfrac{-xy^2}{\sqrt{5-x^2y^2}}$

$z_y = \dfrac{1}{2}(5-x^2y^2)^{-\frac{1}{2}} \cdot (-2x^2y) = \dfrac{-x^2y}{\sqrt{5-x^2y^2}}$

これより，$x=1,\ y=2$ のとき

$z_x = \dfrac{-1 \cdot 2^2}{\sqrt{5-1^2 \cdot 2^2}} = -\dfrac{4}{1} = -4$

$z_y = \dfrac{-1^2 \cdot 2}{\sqrt{5-1^2 \cdot 2^2}} = -\dfrac{2}{1} = -2$

であるから，求める接平面の方程式は

$z - 1 = -4(x-1) - 2(y-2)$

整理して

$z - 1 = -4x + 4 - 2y + 4$

$\boldsymbol{4x + 2y + z = 9}$

（3）　$z_x = \cos(x-y^2) \cdot 1 = \cos(x-y^2)$

$z_y = \cos(x-y^2) \cdot (-2y) = -2y\cos(x-y^2)$

$x=1,\ y=1$ のとき，$z = \sin(1-1^2) = \sin 0 = 0$

また，

$z_x = \cos(1-1^2) = \cos 0 = 1$

$z_y = -2 \cdot 1 \cdot \cos(1-1^2) = -2 \cdot 1 = -2$

であるから，求める接平面の方程式は

$z - 0 = 1(x-1) - 2(y-1)$

整理して

$z = x - 1 - 2y + 2$

$\boldsymbol{x - 2y - z = -1}$

（4）　$z_x = \dfrac{1}{x^2+y^2} \cdot 2x = \dfrac{2x}{x^2+y^2}$

$z_y = \dfrac{1}{x^2+y^2} \cdot 2y = \dfrac{2y}{x^2+y^2}$

$x=1,\ y=0$ のとき，$z = \log(1^2+0^2) = \log 1 = 0$

また，

$z_x = \dfrac{2 \cdot 1}{1^2+0^2} = 2$

$z_y = \dfrac{2 \cdot 0}{1^2+0^2} = 0$

であるから，求める接平面の方程式は

$z - 0 = 2(x-1) - 0(y-0)$

整理して

$z = 2x - 2$

$\boldsymbol{2x - z = 2}$

51（1）　$\dfrac{dx}{dt} = e^t + te^t = (1+t)e^t$

$\dfrac{dy}{dt} = \dfrac{1}{t}$

よって

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$= \boldsymbol{(t+1)e^t\,\dfrac{\partial z}{\partial x} + \dfrac{1}{t}\,\dfrac{\partial z}{\partial y}}$

（2）　$\dfrac{dx}{dt} = \dfrac{1 \cdot (2t+1) - t \cdot 2}{(2t+1)^2}$

$= \dfrac{2t+1-2t}{(2t+1)^2} = \dfrac{1}{(2t+1)^2}$

$\dfrac{dy}{dt} = \dfrac{1(2t+1) - (t+1) \cdot 2}{(2t+1)^2}$

$= \dfrac{2t+1-2t-2}{(2t+1)^2} = -\dfrac{1}{(2t+1)^2}$

よって

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$= \boldsymbol{\dfrac{1}{(2t+1)^2}\,\dfrac{\partial z}{\partial x} - \dfrac{1}{(2t+1)^2}\,\dfrac{\partial z}{\partial y}}$

（3）　$\dfrac{dx}{dt} = \cos t - \sin t$

$\dfrac{dy}{dt} = \cos^2 t + \sin t \cdot (-\sin t) = \cos^2 t - \sin^2 t = \cos 2t$

よって

$\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$= \boldsymbol{(\cos t - \sin t)\,\dfrac{\partial z}{\partial x} + \cos 2t\,\dfrac{\partial z}{\partial y}}$

（4）　$\dfrac{dx}{dt} = -\dfrac{\frac{1}{2}(t+1)^{-\frac{1}{2}} \cdot 1}{(\sqrt{t+1})^2}$

$= -\dfrac{1}{2(t+1)\sqrt{t+1}}$

$\dfrac{dy}{dt} = \dfrac{1}{2}(t+1)^{-\frac{1}{2}} \cdot 1$

$= \dfrac{1}{2\sqrt{t+1}}$

よって

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= -\boldsymbol{\frac{1}{2(t+1)\sqrt{t+1}}\,\frac{\partial z}{\partial x} + \frac{1}{2\sqrt{t+1}}\,\frac{\partial z}{\partial y}}$$

52（1）

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2}$$
$$\frac{\partial z}{\partial y} = \frac{1}{x}$$

また

$$\frac{dx}{dt} = e^t - e^{-t}$$
$$\frac{dy}{dt} = e^t + e^{-t}$$

よって，$\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= -\frac{y}{x^2}(e^t - e^{-t}) + \frac{1}{x}(e^t + e^{-t})$$
$$= -\frac{e^t - e^{-t}}{(e^t + e^{-t})^2}(e^t - e^{-t}) + \frac{1}{e^t + e^{-t}}(e^t + e^{-t})$$
$$= \frac{-(e^t - e^{-t})^2 + (e^t + e^{-t})^2}{(e^t + e^{-t})^2}$$
$$= \frac{-(e^{2t} - 2 + e^{-2t}) + (e^{2t} + 2 + e^{-2t})}{(e^t + e^{-t})^2}$$
$$= \boldsymbol{\frac{4}{(e^t + e^{-t})^2}}$$

（2）

$$\frac{\partial z}{\partial x} = e^{x-y}\cdot 1 = e^{x-y}$$
$$\frac{\partial z}{\partial y} = e^{x-y}\cdot(-1) = -e^{x-y}$$

また

$$\frac{dx}{dt} = \cos t$$
$$\frac{dy}{dt} = -\sin t$$

よって，$\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= e^{x-y}\cdot\cos t - e^{x-y}\cdot(-\sin t)$$
$$= (\cos t + \sin t)e^{x-y}$$
$$= \boldsymbol{(\sin t + \cos t)e^{\sin t - \cos t}}$$

（3）

$$\frac{\partial z}{\partial x} = \frac{1}{x+y}\cdot 1 = \frac{1}{x+y}$$
$$\frac{\partial z}{\partial y} = \frac{1}{x+y}\cdot 1 = \frac{1}{x+y}$$

また

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t+1}}\cdot 1 = \frac{1}{2\sqrt{t+1}}$$
$$\frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}\cdot 1 = \frac{1}{2\sqrt{t-1}}$$

よって，$\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= \frac{1}{x+y}\cdot\frac{1}{2\sqrt{t+1}} + \frac{1}{x+y}\cdot\frac{1}{2\sqrt{t-1}}$$
$$= \frac{\sqrt{t+1} + \sqrt{t-1}}{(x+y)\cdot 2\sqrt{t+1}\sqrt{t-1}}$$
$$= \frac{\sqrt{t+1} + \sqrt{t-1}}{(\sqrt{t+1} + \sqrt{t-1})\cdot 2\sqrt{(t+1)(t-1)}}$$
$$= \boldsymbol{\frac{1}{2\sqrt{t^2-1}}}$$

（4）

$$\frac{\partial z}{\partial x} = \cos(x+2y)\cdot 1 = \cos(x+2y)$$
$$\frac{\partial z}{\partial y} = \cos(x+2y)\cdot 2 = 2\cos(x+2y)$$

また

$$\frac{dx}{dt} = \frac{1}{t}$$
$$\frac{dy}{dt} = -\frac{2}{t^2}$$

よって，$\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= \cos(x+2y)\cdot\frac{1}{t} + 2\cos(x+2y)\cdot\left(-\frac{2}{t^2}\right)$$
$$= \left(\frac{1}{t} - \frac{4}{t^2}\right)\cos(x+2y)$$
$$= \frac{t-4}{t^2}\cos\left(\log t + 2\cdot\frac{2}{t}\right)$$
$$= \boldsymbol{\frac{t-4}{t^2}\cos\left(\log t + \frac{4}{t}\right)}$$

53（1）

$$\frac{\partial x}{\partial u} = 4uv^3,\quad \frac{\partial x}{\partial v} = 6u^2v^2$$
$$\frac{\partial y}{\partial u} = 1,\quad \frac{\partial y}{\partial v} = 3$$

よって

$$z_u = z_x\frac{\partial x}{\partial u} + z_y\frac{\partial y}{\partial u}$$
$$= z_x\cdot 4uv^3 + z_y\cdot 1$$
$$= \boldsymbol{4uv^3 z_x + z_y}$$
$$z_v = z_x\frac{\partial x}{\partial v} + z_y\frac{\partial y}{\partial v}$$
$$= z_x\cdot 6u^2v^2 + z_y\cdot 3$$
$$= \boldsymbol{6u^2v^2 z_x + 3z_y}$$

（2）

$$\frac{\partial x}{\partial u} = 2u,\quad \frac{\partial x}{\partial v} = 2v$$
$$\frac{\partial y}{\partial u} = \frac{1}{v},\quad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

よって

$$z_u = z_x\frac{\partial x}{\partial u} + z_y\frac{\partial y}{\partial u}$$
$$= z_x\cdot 2u + z_y\cdot\frac{1}{v}$$
$$= \boldsymbol{2uz_x + \frac{1}{v}z_y}$$
$$z_v = z_x\frac{\partial x}{\partial v} + z_y\frac{\partial y}{\partial v}$$
$$= z_x\cdot 2v + z_y\cdot\left(-\frac{u}{v^2}\right)$$
$$= \boldsymbol{2vz_x - \frac{u}{v^2}z_y}$$

（3）

$$\frac{\partial x}{\partial u} = \frac{1}{\cos^2\frac{v}{u}}\cdot\left(-\frac{v}{u^2}\right) = -\frac{v}{u^2\cos^2\frac{v}{u}}$$
$$\frac{\partial x}{\partial v} = \frac{1}{\cos^2\frac{v}{u}}\cdot\frac{1}{u} = \frac{1}{u\cos^2\frac{v}{u}}$$
$$\frac{\partial y}{\partial u} = -\sin(u+v)\cdot 1 = -\sin(u+v)$$
$$\frac{\partial y}{\partial v} = -\sin(u+v)\cdot 1 = -\sin(u+v)$$

よって

$$z_u = z_x\frac{\partial x}{\partial u} + z_y\frac{\partial y}{\partial u}$$
$$= z_x\cdot\left(-\frac{v}{u^2\cos^2\frac{v}{u}}\right) + z_y\cdot\{-\sin(u+v)\}$$
$$= \boldsymbol{-\frac{v}{u^2\cos^2\frac{v}{u}}\,z_x - \sin(u+v)\,z_y}$$
$$z_v = z_x\frac{\partial x}{\partial v} + z_y\frac{\partial y}{\partial v}$$
$$= z_x\cdot\frac{1}{u\cos^2\frac{v}{u}} + z_y\cdot\{-\sin(u+v)\}$$
$$= \boldsymbol{\frac{1}{u\cos^2\frac{v}{u}}\,z_x - \sin(u+v)\,z_y}$$

54（1）

$$\frac{\partial z}{\partial x} = 2xy,\quad \frac{\partial z}{\partial y} = x^2$$
$$\frac{\partial x}{\partial u} = 1,\quad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = u$$

よって

$$\begin{aligned}
z_u &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= 2xy \cdot 1 + x^2 \cdot v \\
&= 2xy + x^2 v \\
&= 2(u+v)uv + (u+v)^2 v \\
&= v(u+v)\{2u + (u+v)\} \\
&= \boldsymbol{v(u+v)(3u+v)}
\end{aligned}$$

$$\begin{aligned}
f_v &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \\
&= 2xy \cdot 1 + x^2 \cdot u \\
&= 2xy + x^2 u \\
&= 2(u+v)uv + (x+y)^2 u \\
&= u(u+v)\{2v + (u+v)\} \\
&= \boldsymbol{u(u+v)(u+3v)}
\end{aligned}$$

（2）
$$\frac{\partial z}{\partial x} = \frac{1}{y}, \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2}$$

$$\frac{\partial x}{\partial u} = 2, \quad \frac{\partial x}{\partial v} = 3$$

$$\frac{\partial y}{\partial u} = 3, \quad \frac{\partial y}{\partial v} = -2$$

よって

$$\begin{aligned}
z_u &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \frac{1}{y} \cdot 2 - \frac{x}{y^2} \cdot 3 \\
&= \frac{2}{y} - \frac{3x}{y^2} \\
&= \frac{2y - 3x}{y^2} \\
&= \frac{2(3u-2v) - 3(2u+3v)}{(3u-2v)^2} \\
&= \frac{6u - 4v - 6u - 9v}{(3u-2v)^2} \\
&= \boldsymbol{-\frac{13v}{(3u-2v)^2}}
\end{aligned}$$

$$\begin{aligned}
z_v &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \frac{1}{y} \cdot 3 - \frac{x}{y^2} \cdot (-2) \\
&= \frac{3}{y} + \frac{2x}{y^2} \\
&= \frac{3y + 2x}{y^2} \\
&= \frac{3(3u-2v) + 2(2u+3v)}{(3u-2v)^2} \\
&= \frac{9u - 6v + 4u + 6v}{(3u-2v)^2} \\
&= \boldsymbol{\frac{13u}{(3u-2v)^2}}
\end{aligned}$$

（3）
$$\frac{\partial z}{\partial x} = 2 \cdot \frac{1}{2\sqrt{x+y}} \cdot 1 = \frac{1}{\sqrt{x+y}}$$

$$\frac{\partial z}{\partial y} = 2 \cdot \frac{1}{2\sqrt{x+y}} \cdot 1 = \frac{1}{\sqrt{x+y}}$$

$$\frac{\partial x}{\partial u} = \cos(2u+v) \cdot 2 = 2\cos(2u+v)$$

$$\frac{\partial x}{\partial v} = \cos(2u+v) \cdot 1 = \cos(2u+v)$$

$$\frac{\partial y}{\partial u} = -\sin(u-2v) \cdot 1 = -\sin(u-2v)$$

$$\frac{\partial y}{\partial v} = -\sin(u-2v) \cdot (-2) = 2\sin(u-2v)$$

よって

$$\begin{aligned}
z_u &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \frac{1}{\sqrt{x+y}} \cdot 2\cos(2u+v) + \frac{1}{\sqrt{x+y}} \cdot \{-\sin(u-2v)\} \\
&= \frac{2\cos(2u+v) - \sin(u-2v)}{\sqrt{x+y}} \\
&= \boldsymbol{\frac{2\cos(2u+v) - \sin(u-2v)}{\sqrt{\sin(2u+v) + \cos(u-2v)}}}
\end{aligned}$$

$$\begin{aligned}
z_v &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \frac{1}{\sqrt{x+y}} \cdot \cos(2u+v) + \frac{1}{\sqrt{x+y}} \cdot 2\sin(u-2v) \\
&= \frac{\cos(2u+v) + 2\sin(u-2v)}{\sqrt{x+y}} \\
&= \boldsymbol{\frac{\cos(2u+v) + 2\sin(u-2v)}{\sqrt{\sin(2u+v) + \cos(u-2v)}}}
\end{aligned}$$

（4）
$$\frac{\partial z}{\partial x} = 2x\log y, \quad \frac{\partial z}{\partial y} = x^2 \cdot \frac{1}{y} = \frac{x^2}{y}$$

$$\frac{\partial x}{\partial u} = 2, \quad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = v, \quad \frac{\partial y}{\partial v} = u$$

よって

$$\begin{aligned}
z_u &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= 2x\log y \cdot 2 + \frac{x^2}{y} \cdot v \\
&= 4(2u+v)\log(uv) + \frac{(2u+v)^2}{uv} \cdot v \\
&= \boldsymbol{4(2u+v)\log(uv) + \frac{(2u+v)^2}{u}}
\end{aligned}$$

$$\begin{aligned}
z_v &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= 2x\log y \cdot 1 + \frac{x^2}{y} \cdot u \\
&= 2(2u+v)\log(uv) + \frac{(2u+v)^2}{uv} \cdot u \\
&= \boldsymbol{2(2u+v)\log(uv) + \frac{(2u+v)^2}{v}}
\end{aligned}$$

∎