3章　重積分

§1　2重積分 (p.27〜p.)

BASIC

108　$x^2+y^2+z^2=9$ より，$z=\pm\sqrt{9-x^2-y^2}$

$z \geqq 0$ なので，$z=\sqrt{9-x^2-y^2}$

また，領域 D を

$D=\{(x,y)\mid x^2+y^2 \leqq 4\}$ とすれば

$$\boldsymbol{V=\iint_D \sqrt{9-x^2-y^2}\,dx\,dy}$$

109　$2 \leqq x \leqq 3$ より，$\boldsymbol{p=2,\ q=3}$

$0 \leqq y \leqq 1$ より，$\boldsymbol{r=0,\ s=1}$

$$\begin{aligned}
\text{与式} &= \int_2^3 \left\{\int_0^1 (x^2y-y^3)\,dy\right\} dx \\
&= \int_2^3 \left[\frac{1}{2}x^2y^2-\frac{1}{4}y^4\right]_0^1 dx \\
&= \int_2^3 \left(\frac{1}{2}x^2-\frac{1}{4}-0\right) dx \\
&= \frac{1}{4}\int_2^3 (2x^2-1)\,dx \\
&= \frac{1}{4}\left[\frac{2}{3}x^3-x\right]_2^3 \\
&= \frac{1}{4}\left\{\left(\frac{2}{3}\cdot 3^3-3\right)-\left(\frac{2}{3}\cdot 2^3-2\right)\right\} \\
&= \frac{1}{4}\left(18-3-\frac{16}{3}+2\right) \\
&= \frac{1}{4}\left(17-\frac{16}{3}\right) \\
&= \frac{1}{4}\cdot\frac{51-16}{3} \\
&= \frac{1}{4}\cdot\frac{35}{3} = \boldsymbol{\frac{35}{12}}
\end{aligned}$$

110（1）

$$\begin{aligned}
\text{与式} &= \int_0^1 \left\{\int_1^3 (xy^2+y)\,dy\right\} dx \\
&= \int_0^1 \left[\frac{1}{3}xy^3+\frac{1}{2}y^2\right]_1^3 dx \\
&= \int_0^1 \left\{\left(\frac{1}{3}x\cdot 3^3+\frac{1}{2}\cdot 3^2\right)\right. \\
&\qquad \left. -\left(\frac{1}{3}x\cdot 1^3+\frac{1}{2}\cdot 1^2\right)\right\} dx \\
&= \int_0^1 \left(9x+\frac{9}{2}-\frac{1}{3}x-\frac{1}{2}\right) dx \\
&= \int_0^1 \left(\frac{27-1}{3}x+4\right) dx \\
&= \int_0^1 \left(\frac{26}{3}x+4\right) dx \\
&= \left[\frac{13}{3}x^2+4x\right]_0^1 \\
&= \frac{13}{3}+4 \\
&= \frac{13+12}{3} = \boldsymbol{\frac{25}{3}}
\end{aligned}$$

〔別解〕

$$\begin{aligned}
\text{与式} &= \int_1^3 \left\{\int_0^1 (xy^2+y)\,dx\right\} dy \\
&= \int_1^3 \left[\frac{1}{2}y^2x^2+yx\right]_0^1 dy \\
&= \int_1^3 \left\{\left(\frac{1}{2}y^2\cdot 1^2+y\cdot 1\right)-0\right\} dy \\
&= \int_1^3 \left(\frac{1}{2}y^2+y\right) dy \\
&= \left[\frac{1}{6}y^3+\frac{1}{2}y^2\right]_1^3 \\
&= \left(\frac{1}{6}\cdot 3^3+\frac{1}{2}\cdot 3^2\right)-\left(\frac{1}{6}\cdot 1^3+\frac{1}{2}\cdot 1^2\right) \\
&= \frac{9}{2}+\frac{9}{2}-\frac{1}{6}-\frac{1}{2} \\
&= \frac{27+27-1-3}{6} \\
&= \frac{50}{6} = \boldsymbol{\frac{25}{3}}
\end{aligned}$$

（2）

$$\begin{aligned}
\text{与式} &= \int_0^1 \left\{\int_1^2 e^{2x+y}\,dy\right\} dx \\
&= \int_0^1 \left[e^{2x+y}\right]_1^2 dx \\
&= \int_0^1 (e^{2x+2}-e^{2x+1})\,dx \\
&= \int_0^1 (e^2-e)e^{2x}\,dx \\
&= (e^2-e)\left[\frac{1}{2}e^{2x}\right]_0^1 \\
&= \frac{1}{2}e(e-1)(e^2-e^0) \\
&= \frac{1}{2}e(e-1)(e^2-1) \\
&= \frac{1}{2}e(e-1)(e+1)(e-1) \\
&= \boldsymbol{\frac{1}{2}e(e+1)(e-1)^2}
\end{aligned}$$

〔別解〕

$$\begin{aligned}
\text{与式} &= \int_1^2 \left\{\int_0^1 e^{2x+y}\,dx\right\} dy \\
&= \int_1^2 \left[\frac{1}{2}e^{2x+y}\right]_0^1 dy \\
&= \frac{1}{2}\int_1^2 (e^{2+y}-e^y)\,dy \\
&= \frac{1}{2}\int_1^2 (e^2-1)e^y\,dy \\
&= \frac{1}{2}(e^2-1)\left[e^y\right]_1^2 \\
&= \frac{1}{2}(e+1)(e-1)(e^2-e^1) \\
&= \frac{1}{2}e(e+1)(e-1)(e-1) \\
&= \boldsymbol{\frac{1}{2}e(e+1)(e-1)^2}
\end{aligned}$$

（3）　与式 $= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} \sin(x-y)\,dy \right\} dx$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{-1} \cdot \{-\cos(x-y)\} \right]_0^{\frac{\pi}{2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\cos(x-y) \right]_0^{\frac{\pi}{2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \left\{ \cos\left(x-\frac{\pi}{2}\right) - \cos(x-0) \right\} dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x \Big]_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (-\cos 0 - \sin 0)$$

$$= -0 - 1 + 1 + 0 = \mathbf{0}$$

〔別解〕

与式 $= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} \sin(x-y)\,dx \right\} dy$

$$= \int_0^{\frac{\pi}{2}} \Big[-\cos(x-y) \Big]_0^{\frac{\pi}{2}} dy$$

$$= -\int_0^{\frac{\pi}{2}} \left\{ \cos\left(\frac{\pi}{2}-y\right) - \cos(0-y) \right\} dy$$

$$= -\int_0^{\frac{\pi}{2}} (\sin y - \cos y)\,dy$$

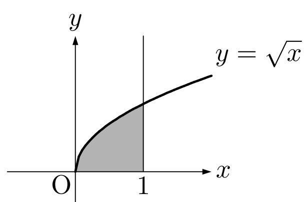
$$= -\Big[-\cos y - \sin y \Big]_0^{\frac{\pi}{2}}$$

$$= \Big[\cos y + \sin y \Big]_0^{\frac{\pi}{2}}$$

$$= \left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (\cos 0 + \sin 0)$$

$$= 1 + 0 - 1 - 0 = \mathbf{0}$$

111（1）　領域を図示すると



与式 $= \int_0^1 \left\{ \int_0^{\sqrt{x}} xy\,dy \right\} dx$

$$= \int_0^1 \left[\frac{1}{2}xy^2 \right]_0^{\sqrt{x}} dx$$

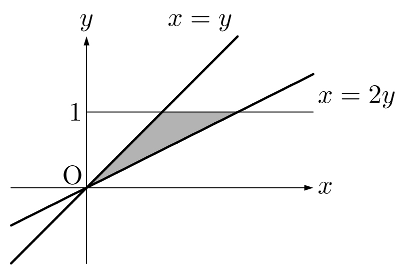
$$= \frac{1}{2}\int_0^1 \{x \cdot (\sqrt{x})^2 - 0\}\,dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx$$

$$= \frac{1}{2}\left[\frac{1}{3}x^3 \right]_0^1$$

$$= \frac{1}{2} \cdot \frac{1}{3} = \mathbf{\frac{1}{6}}$$

（2）　領域を図示すると



与式 $= \int_0^1 \left\{ \int_y^{2y} (x-y)\,dx \right\} dy$

$$= \int_0^1 \left[\frac{1}{2}x^2 - yx \right]_y^{2y} dy$$

$$= \int_0^1 \left\{ \left(\frac{1}{2} \cdot (2y)^2 - y \cdot 2y \right) - \left(\frac{1}{2} \cdot y^2 - y \cdot y \right) \right\} dy$$

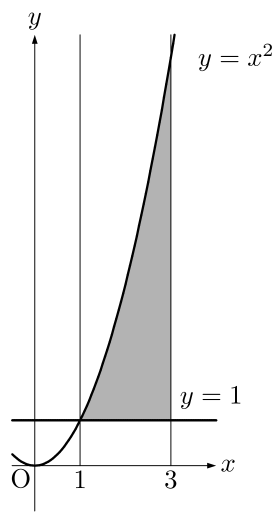
$$= \int_0^1 \left(2y^2 - 2y^2 - \frac{1}{2}y^2 + y^2 \right) dy$$

$$= \int_0^1 \frac{1}{2}y^2\,dy$$

$$= \left[\frac{1}{6}y^3 \right]_0^1$$

$$= \frac{1}{6} \cdot 1^3 = \mathbf{\frac{1}{6}}$$

（3）　領域を図示すると



与式 $= \int_1^3 \left\{ \int_1^{x^2} \frac{x}{y^2}\,dy \right\} dx$

$$= \int_1^3 \left[-\frac{x}{y} \right]_1^{x^2} dx$$

$$= -\int_1^3 \left(\frac{x}{x^2} - \frac{x}{1} \right) dx$$

$$= -\int_1^3 \left(\frac{1}{x} - x \right) dx$$

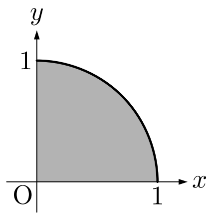
$$= -\left[\log|x| - \frac{1}{2}x^2 \right]_1^3$$

$$= -\left\{ \left(\log 3 - \frac{1}{2} \cdot 3^2 \right) - \left(\log 1 - \frac{1}{2} \cdot 1^2 \right) \right\}$$

$$= -\left(\log 3 - \frac{9}{2} - 0 + \frac{1}{2} \right)$$

$$= -\left(\log 3 - \frac{8}{2} \right) = \mathbf{4 - \log 3}$$

112（1）　領域を図示すると



$x^2+y^2 \leqq 1$ より，$y^2 \leqq 1-x^2$

これより，$-\sqrt{1-x^2} \leqq y \leqq \sqrt{1-x^2}$

また，$1-x^2 \geqq 0$ より，$-1 \leqq x \leqq 1$

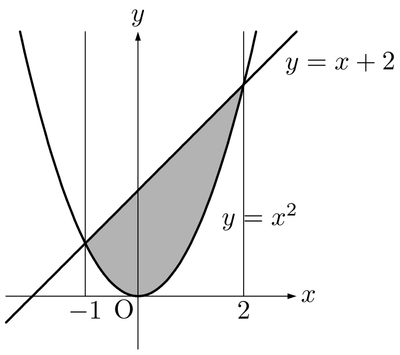
以上より，領域 D は次の不等式で表すことができる．

$0 \leqq x \leqq 1,\ \ 0 \leqq y \leqq \sqrt{1-x^2}$

したがって

$$\begin{aligned}
与式 &= \int_0^1 \left\{ \int_0^{\sqrt{1-x^2}} x^2 y\, dy \right\} dx \\
&= \int_0^1 \left[\frac{1}{2}x^2y^2 \right]_0^{\sqrt{1-x^2}} dx \\
&= \frac{1}{2}\int_0^1 \{x^2 \cdot (\sqrt{1-x^2})^2 - 0\}\, dx \\
&= \frac{1}{2}\int_0^1 x^2(1-x^2)\, dx \\
&= \frac{1}{2}\int_0^1 (x^2-x^4)\, dx \\
&= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 \\
&= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} - 0 \right) \\
&= \frac{1}{2} \cdot \frac{5-3}{15} \\
&= \frac{1}{2} \cdot \frac{2}{15} = \boldsymbol{\frac{1}{15}}
\end{aligned}$$

（2）　領域を図示すると



グラフの交点は，$x^2 = x+2$ を解いて，$x=-1,\ 2$

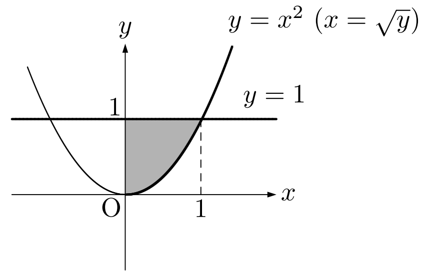
よって，領域 D は次の不等式で表すことができる．

$-1 \leqq x \leqq 2,\ \ x^2 \leqq y \leqq x+2$

したがって

$$\begin{aligned}
与式 &= \int_{-1}^2 \left\{ \int_{x^2}^{x+2} (x-2y)\, dy \right\} dx \\
&= \int_{-1}^2 \left[xy - y^2 \right]_{x^2}^{x+2} dx \\
&= \int_{-1}^2 \{\{x(x+2)-(x+2)^2\} \\
&\qquad\qquad -\{x\cdot x^2 - (x^2)^2\}\}\, dx \\
&= \int_{-1}^2 \{x^2+2x-(x^2+4x+4)) - x^3 + x^4\}\, dx \\
&= \int_{-1}^2 (x^4-x^3-2x-4)\, dx \\
&= \left[\frac{1}{5}x^5 - \frac{1}{4}x^4 - x^2 - 4x \right]_{-1}^2 \\
&= \left(\frac{1}{5}\cdot 2^5 - \frac{1}{4}\cdot 2^4 - 2^2 - 4\cdot 2 \right) \\
&\quad - \left\{ \frac{1}{5}\cdot(-1)^5 - \frac{1}{4}\cdot(-1)^4 - (-1)^2 - 4\cdot(-1) \right\} \\
&= \frac{32}{5} - 4 - 4 - 8 - \left(-\frac{1}{5} - \frac{1}{4} - 1 + 4 \right) \\
&= \frac{33}{5} + \frac{1}{4} - 19 \\
&= \frac{132+5-380}{20} = \boldsymbol{-\frac{243}{20}}
\end{aligned}$$

113（1）　領域を図示すると



$y = x^2$ より，$x = \pm\sqrt{y}$

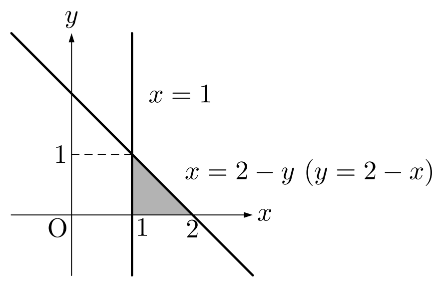
よって，領域 D は次の不等式で表すことができる．

$0 \leqq x \leqq \sqrt{y},\ \ 0 \leqq y \leqq 1$

したがって

$$与式 = \boldsymbol{\int_0^1 \left\{ \int_0^{\sqrt{y}} f(x,\ y)\, dx \right\} dy}$$

（2）　領域を図示すると



$x = 2-y$ より，$y = 2-x$

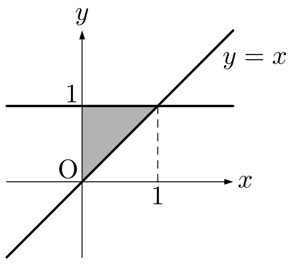
よって，領域 D は次の不等式で表すことができる．

$1 \leqq x \leqq 2,\ \ 0 \leqq y \leqq 2-x$

したがって

$$与式 = \boldsymbol{\int_1^2 \left\{ \int_0^{2-x} f(x,\ y)\, dy \right\} dx}$$

114　　$0 \leqq x \leqq 1,\ \ x \leqq y \leqq 1$ であるから，領域は図のようになる．



この領域は，$0 \leqq x \leqq y,\ \ 0 \leqq y \leqq 1$ と表せるので

$$\text{与式} = \int_0^1 \left\{ \int_0^y \sqrt{y^2+1}\,dx \right\} dy$$
$$= \int_0^1 \sqrt{y^2+1} \Big[\ x\ \Big]_0^y dy$$
$$= \int_0^1 y\sqrt{y^2+1}\,dy$$

$\sqrt{y^2+1} = t$ とおくと，$y^2+1 = t^2$

$2y\,dy = 2t\,dt$ より，$y\,dy = t\,dt$

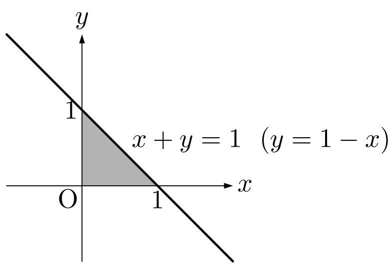
また，y と t の対応は

y	0	$\to$	1
t	1	$\to$	$\sqrt{2}$

よって

$$\text{与式} = \int_1^{\sqrt{2}} t \cdot t\,dt$$
$$= \int_1^{\sqrt{2}} t^2\,dt$$
$$= \left[\frac{1}{3} t^3 \right]_1^{\sqrt{2}}$$
$$= \frac{1}{3}\{(\sqrt{2})^3 - 1^3\}$$
$$= \boldsymbol{\frac{1}{3}(2\sqrt{2}-1)}$$

115 領域を図示すると



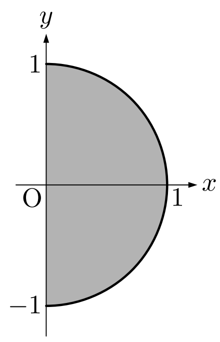
求める体積を V とする．$x+y=1$ より，$y = 1-x$ であるから，領域は次の不等式で表すことができる．

$0 \leqq x \leqq 1,\ \ 0 \leqq y \leqq 1-x$

この領域内で $z = x^2+2y^2 \geqq 0$ なので

$$V = \int_0^1 \left\{ \int_0^{1-x} (x^2+2y^2)\,dy \right\} dx$$
$$= \int_0^1 \left[x^2 y + \frac{2}{3} y^3 \right]_0^{1-x} dx$$
$$= \int_0^1 \left\{ x^2(1-x) + \frac{2}{3}(1-x)^3 - 0 \right\} dx$$
$$= \frac{1}{3}\int_0^1 \left\{ 3x^2(1-x) + 2(1-x)^3 \right\} dx$$
$$= \frac{1}{3}\int_0^1 (3x^2 - 3x^3 + 2 - 6x + 6x^2 - 2x^3)\,dx$$
$$= \frac{1}{3}\int_0^1 (-5x^3 + 9x^2 - 6x + 2)\,dx$$
$$= \frac{1}{3}\left[-\frac{5}{4}x^4 + 3x^3 - 3x^2 + 2x \right]_0^1$$
$$= \frac{1}{3}\left(-\frac{5}{4} + 3 - 3 + 2 \right)$$
$$= \frac{1}{3} \cdot \frac{-5+8}{4} = \frac{1}{3} \cdot \frac{3}{4} = \boldsymbol{\frac{1}{4}}$$

116 領域を図示すると



$x^2+y^2 = 1$ より，$x = \pm\sqrt{1-y^2}$ であるから，この領域は次の不等式で表すことができる．

$0 \leqq x \leqq \sqrt{1-y^2},\quad -1 \leqq y \leqq 1$

この領域内で $z = 2x \geqq 0$ なので

$$V = \int_{-1}^1 \left\{ \int_0^{\sqrt{1-y^2}} 2x\,dx \right\} dy$$
$$= \int_{-1}^1 \Big[x^2 \Big]_0^{\sqrt{1-y^2}} dy$$
$$= \int_{-1}^1 \{(\sqrt{1-y^2})^2 - 0\}\,dy$$
$$= \int_{-1}^1 (1-y^2)\,dy$$
$$= 2\int_0^1 (1-y^2)\,dy$$
$$= 2\left[y - \frac{1}{3}y^3 \right]_0^1$$
$$= 2\left(1 - \frac{1}{3} - 0 \right)$$
$$= 2 \cdot \frac{2}{3} = \boldsymbol{\frac{4}{3}}$$

∎

CHECK