定積分の $\frac{1}{6}$ 公式

$$\int_{\alpha}^{\beta} (x-\alpha)(x-\beta)\,dx = -\frac{1}{6}(\beta-\alpha)^3$$

証明 1

$$\begin{aligned}
\int_{\alpha}^{\beta} (x-\alpha)(x-\beta)\,dx &= \int_{\alpha}^{\beta} \{x^2 - (\alpha+\beta) + \alpha\beta\}\,dx \\
&= \left[\frac{1}{3}x^3 - \frac{1}{2}(\alpha+\beta)x^2 + \alpha\beta x\right]_{\alpha}^{\beta} \\
&= \frac{1}{3}(\beta^3-\alpha^3) - \frac{1}{2}(\alpha+\beta)(\beta^2-\alpha^2) + \alpha\beta(\beta-\alpha) \\
&= \frac{1}{3}(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2) - \frac{1}{2}(\beta-\alpha)(\alpha+\beta)^2 + \alpha\beta(\beta-\alpha) \\
&= \frac{1}{6}(\beta-\alpha)\{2(\beta^2+\alpha\beta+\beta^2) - 3(\alpha+\beta)^2 + 6\alpha\beta\} \\
&= \frac{1}{6}(\beta-\alpha)\{2\beta^2+2\alpha\beta+2\beta^2-3\alpha^2-6\alpha\beta-3\beta^2+6\alpha\beta\} \\
&= \frac{1}{6}(\beta-\alpha)(-\beta^2+2\alpha\beta-\alpha^2) \\
&= -\frac{1}{6}(\beta-\alpha)(\beta^2-2\alpha\beta+\alpha^2) \\
&= -\frac{1}{6}(\beta-\alpha)(\beta-\alpha)^2 \\
&= -\frac{1}{6}(\beta-\alpha)^3 \quad \blacksquare
\end{aligned}$$

証明 2

$$\begin{aligned}
(x-\alpha)(x-\beta) &= (x-\alpha)\{(x-\alpha)+(\alpha-\beta)\} \\
&= (x-\alpha)^2 + (\alpha-\beta)(x-\alpha)
\end{aligned}$$

よって

$$\begin{aligned}
\int_{\alpha}^{\beta} (x-\alpha)(x-\beta)\,dx &= \int_{\alpha}^{\beta} \{(x-\alpha)^2 + (\alpha-\beta)(x-\alpha)\}\,dx \\
&= \int_{\alpha}^{\beta} (x-\alpha)^2\,dx + \int_{\alpha}^{\beta} (\alpha-\beta)(x-\alpha)\,dx \\
&= \left[\frac{1}{3}(x-\alpha)^3\right]_{\alpha}^{\beta} + (\alpha-\beta)\left[\frac{1}{2}(x-\alpha)^2\right]_{\alpha}^{\beta} \\
&= \frac{1}{3}(\beta-\alpha)^3 + (\alpha-\beta)\cdot\frac{1}{2}(\beta-\alpha)^2 \\
&= \frac{1}{3}(\beta-\alpha)^3 - \frac{1}{2}(\beta-\alpha)^3 \\
&= -\frac{1}{6}(\beta-\alpha)^3 \quad \blacksquare
\end{aligned}$$

例題 次の定積分を求めなさい。

（1）$\displaystyle\int_{-2}^{1}(x^2+x-2)\,dx$

〔解答〕

$$
\begin{aligned}
\text{与式} &= \int_{-2}^{1}(x+2)(x-1)\,dx \\
&= -\frac{1}{6}\{1-(-2)\}^3 \\
&= -\frac{1}{6}\cdot 3^3 = \boldsymbol{-\frac{9}{2}}
\end{aligned}
$$

（2）$\displaystyle\int_{-1}^{3}(-x^2+2x+3)\,dx$

〔解答〕

$$
\begin{aligned}
\text{与式} &= -\int_{-1}^{3}(x^2-2x-3)\,dx \\
&= -\int_{-1}^{3}(x+1)(x-3)\,dx \\
&= -\left\{-\frac{1}{6}\{3-(-1)\}^3\right\} \\
&= \frac{1}{6}\cdot 4^3 = \boldsymbol{\frac{32}{3}}
\end{aligned}
$$

（3）$\displaystyle\int_{-\frac{1}{2}}^{2}(2x^2-3x-2)\,dx$

〔解答〕

$$
\begin{aligned}
\text{与式} &= \int_{-\frac{1}{2}}^{2}(2x+1)(x-2)\,dx \\
&= \int_{-\frac{1}{2}}^{2}2\left(x+\frac{1}{2}\right)(x-2)\,dx \\
&= 2\int_{-\frac{1}{2}}^{2}\left(x+\frac{1}{2}\right)(x-2)\,dx \\
&= 2\left\{-\frac{1}{6}\left\{2-\left(-\frac{1}{2}\right)\right\}^3\right\} \\
&= -\frac{1}{3}\left(\frac{5}{2}\right)^3 = -\frac{1}{3}\cdot\frac{125}{8} = \boldsymbol{-\frac{125}{24}}
\end{aligned}
$$

（4）$\displaystyle\int_{1-\sqrt{3}}^{1+\sqrt{3}}(x^2-2x-2)\,dx$

〔解答〕

$x^2-2x-2=0$ を解くと，$x=1\pm\sqrt{3}$

よって

$$
\begin{aligned}
\text{与式} &= \int_{1-\sqrt{3}}^{1+\sqrt{3}}\{x-(1-\sqrt{3})\}\{x-(1+\sqrt{3})\}\,dx \\
&= -\frac{1}{6}\{(1+\sqrt{3})-(1-\sqrt{3})\}^3 \\
&= -\frac{1}{6}\cdot(2\sqrt{3})^3 = \boldsymbol{-4\sqrt{3}}
\end{aligned}
$$