INTRODUCTION TO A THEORY OF \(b\)-FUNCTIONS

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We give an introduction to a theory of \(b\)-functions, i.e. Bernstein-Sato polynomials. After reviewing some facts from \(D\)-modules, we introduce \(b\)-functions including the one for arbitrary ideals of the structure sheaf. We explain the relation with singularities, multiplier ideals, etc., and calculate the \(b\)-functions of monomial ideals and also of hyperplane arrangements in certain cases.

1. D-modules.

1.1. Let \(X\) be a complex manifold or a smooth algebraic variety over \(\mathbb{C}\). Let \(\mathcal{D}_X\) be the ring of partial differential operators. A local section of \(\mathcal{D}_X\) is written as

\[
\sum_{\nu \in \mathbb{N}_n} a_\nu \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \in \mathcal{D}_X
\]

with \(a_\nu \in \mathcal{O}_X\), where \(\partial_i = \partial/\partial x_i\) with \((x_1, \ldots, x_n)\) a local coordinate system.

Let \(F\) be the filtration by the order of operators i.e.

\[
F_p \mathcal{D}_X = \{ \sum_{|\nu| \leq p} a_\nu \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} \},
\]

where \(|\nu| = \sum i \nu_i\). Let \(\xi_i = \text{Gr}^F_1 \partial_i \in \text{Gr}^F_1 \mathcal{D}_X\). Then

\[
\text{Gr}^F \mathcal{D}_X := \bigoplus_p \text{Gr}^F_p \mathcal{D}_X = \bigoplus_p \text{Sym}^p \Theta_X (= \mathcal{O}_X[\xi_1, \ldots, \xi_n] \text{ locally}),
\]

\[
\text{Spec}_X \text{Gr}^F \mathcal{D}_X = T^*X.
\]

1.2 Definition. We say that a left \(\mathcal{D}_X\)-module \(M\) is coherent if it has locally a finite presentation

\[
\bigoplus \mathcal{D}_X \to \bigoplus \mathcal{D}_X \to M \to 0.
\]

1.3. Remark. A left \(\mathcal{D}_X\)-module \(M\) is coherent if and only if it is quasi-coherent over \(\mathcal{O}_X\) and locally finitely generated over \(\mathcal{D}_X\). (It is known that \(\text{Gr}^F \mathcal{D}_X\) is a noetherian ring, i.e. an increasing sequence of locally finitely generated \(\text{Gr}^F \mathcal{D}_X\)-submodules of a coherent \(\text{Gr}^F \mathcal{D}_X\)-module is locally stationary.)

1.4. Definition. A filtration \(F\) on a left \(\mathcal{D}_X\)-module \(M\) is good if \((M, F)\) is a coherent filtered \(\mathcal{D}_X\)-module, i.e. if \(F_p \mathcal{D}_X F_q M \subseteq M_{p+q}\) and \(\text{Gr}^F M := \bigoplus_p \text{Gr}^F_p M\) is coherent over \(\text{Gr}^F \mathcal{D}_X\).

1.5. Remark. A left \(\mathcal{D}_X\)-module \(M\) is coherent if and only if it has a good filtration locally.

1.6. Characteristic varieties. For a coherent left $\mathcal{D}_X$-module $M$, we define the characteristic variety $CV(M)$ by
\begin{equation}
CV(M) = \text{Supp} \text{Gr}^F M \subset T^*M,
\end{equation}
taking locally a good filtration $F$ of $M$.

1.7. Remark. The above definition is independent of the choice of $F$. If $M = \mathcal{D}_X/I$ for a coherent left ideal $I$ of $\mathcal{D}_X$, take $P_i \in F_k[I]$ such that the $\rho_i := \text{Gr}^F_k P_i$ generate $\text{Gr}^F[I] \text{ over Gr}^F \mathcal{D}_X$. Then $CV(M)$ is defined by the $\rho_i \in \mathcal{O}_X[\xi_1, \ldots, \xi_n]$.

1.8. Theorem (Sato, Kawai, Kashiwara [39], Bernstein [2]). We have the inequality $\dim CV(M) \geq \dim X$. (More precisely, $CV(M)$ is involutive, see [39].)

1.9. Definition. We say that a left $\mathcal{D}_X$-module $M$ is holonomic if it is coherent and $\dim CV(M) = \dim X$.

2. De Rham functor.

2.1. Definition. For a left $\mathcal{D}_X$-module $M$, we define the de Rham functor $\text{DR}(M)$ by
\begin{equation}
M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^{\dim X} \otimes_{\mathcal{O}_X} M,
\end{equation}
where the last term is put at the degree 0. In the algebraic case, we use analytic sheaves or replace $M$ with the associated analytic sheaf $M^\text{an} := M \otimes_{\mathcal{O}_X} \mathcal{O}_X^\text{an}$ in case $M$ is algebraic (i.e. $M$ is an $\mathcal{O}_X$-module with $\mathcal{O}_X$ algebraic).

2.2. Perverse sheaves. Let $D^b_c(X, \mathbb{C})$ be the derived category of bounded complexes of $\mathcal{C}_X$-modules $K$ with $H^j K$ constructible. (In the algebraic case we use analytic topology for the sheaves although we use Zariski topology for constructibility.) Then the category of perverse sheaves $\text{Perv}(X, \mathbb{C})$ is a full subcategory of $D^b_c(X, \mathbb{C})$ consisting of $K$ such that
\begin{equation}
\dim \text{Supp} H^{-j}K \leq j, \quad \dim \text{Supp} H^{-j}\mathcal{D} K \leq j,
\end{equation}
where $\mathcal{D} K := \mathcal{R}\text{Hom}(K, \mathbb{C}[2 \dim X])$ is the dual of $K$, and $H^j K$ is the $j$-th cohomology sheaf of $K$.

2.3. Theorem (Beilinson, Bernstein, Deligne [1]). $\text{Perv}(X, \mathbb{C})$ is an abelian category.

2.4. Theorem (Kashiwara). If $M$ is holonomic, then $\text{DR}(M)$ is a perverse sheaf.

Outline of proof. By Kashiwara [19], we have $\text{DR}(M) \in D^b_c(X, \mathbb{C})$, and the first condition of (2.2.1) is verified. Then the assertion follows from the commutativity of the dual $\mathcal{D}$ and the de Rham functor $\text{DR}$.

2.5. Example. $\text{DR}(\mathcal{O}_X) = \mathbb{C}_X[\dim X]$.

2.6. Direct images. For a closed immersion $i : X \to Y$ such that $X$ is defined by $x_i = 0$ in $Y$ for $1 \leq i \leq r$, define the direct image of left $\mathcal{D}_X$-modules $M$ by
\begin{equation}
i_+M := M[\partial_1, \ldots, \partial_r].
\end{equation}
(Globally there is a twist by a line bundle.) For a projection \( p : X \times Y \to Y \), define
\[
p_+ M = \mathbb{R} p_+ \text{DR}_X(M).
\]
In general, \( f_+ = p_+ i_+ \) using \( f = pi \) with \( i \) graph embedding. See [4] for details.

2.7. Regular holonomic D-modules. Let \( M \) be a holonomic \( \mathcal{D}_X \)-module with support \( Z \), and \( U \) a Zariski-open of \( Z \) such that \( \text{DR}(M)|_U \) is a local system up to a shift. Then \( M \) is regular if and only if there exists locally a divisor \( D \) on \( X \) containing \( Z \setminus U \) and such that \( M(\ast D) \) is the direct image of a regular holonomic \( \mathcal{D} \)-module ‘of Deligne-type’ (see [11]) on a desingularization of \((Z, Z \cap D)\), and \( \text{Ker}(M \to M(\ast D)) \) is regular holonomic (by induction on \( \dim \text{Supp} M \)).

Note that the category \( M_{rh}(\mathcal{D}_X) \) of regular holonomic \( \mathcal{D}_X \)-modules is stable by subquotients and extensions in the category \( M_b(\mathcal{D}_X) \) of holonomic \( \mathcal{D}_X \)-modules.

2.8. Theorem (Kashiwara-Kawai [24], [22], Mebkhout [28]).
(i) The structure sheaf \( \mathcal{O}_X \) is regular holonomic.
(ii) The functor \( \text{DR} \) induces an equivalence of categories
\[
\text{DR} : M_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \text{Perv}(X, \mathbb{C}).
\]
(See [4] for the algebraic case.)

3. \( b \)-Functions.

3.1. Definition. Let \( f \) be a holomorphic function on \( X \), or \( f \in \Gamma(X, \mathcal{O}_X) \) in the algebraic case. Then we have
\[
\mathcal{D}_X[s] f^s \subset \mathcal{O}_X[\frac{1}{s}][s] f^s \quad \text{where } \partial_i f^s = s(\partial_i f) f^{s-1},
\]
and \( b_f(s) \) is the monic polynomial of the least degree satisfying
\[
b_f(s) f^s = P(x, \partial, s) f^{s+1} \quad \text{in } \mathcal{O}_X[\frac{1}{s}][s] f^s,
\]
with \( P(x, \partial, s) \in \mathcal{D}_X[s] \). Locally, it is the minimal polynomial of the action of \( s \) on
\[
\mathcal{D}_X[s] f^s / \mathcal{D}_X[s] f^{s+1}.
\]
We define \( b_{f,x}(s) \) replacing \( \mathcal{D}_X \) with \( \mathcal{D}_{X,x} \).

3.2. Theorem (Sato [38], Bernstein [2], Bjork [3]). The \( b \)-function exists at least locally, and exists globally in the case \( X \) affine variety with \( f \) algebraic.

3.3. Observation. Let \( i_f : X \to \widetilde{X} := X \times \mathbb{C} \) be the graph embedding. Then there are canonical isomorphisms
\[
\widetilde{M} := i_f_+ \mathcal{O}_X = \mathcal{O}_X[\partial_i] \delta(f - t) = \mathcal{O}_{X \times \mathbb{C}}[\frac{1}{t - i}] / \mathcal{O}_{X \times \mathbb{C}},
\]
where the action of \( \partial_i \) on \( \delta(f - t) (= \frac{1}{t - i}) \) is given by
\[
\partial_i \delta(f - t) = -\delta(f + t) \partial_i \delta(f - t).
\]
Moreover, \( f^s \) is canonically identified with \( \delta(f - t) \) setting \( s = -\partial_i t \), and we have a canonical isomorphism as \( \mathcal{D}_{X[s]} \)-modules
\[
\mathcal{D}_X[s] f^s = \mathcal{D}_X[s] \delta(f - t).
\]
3.4. V-filtration. We say that $V$ is a filtration of Kashiwara-Malgrange if $V$ is exhaustive, separated, and satisfies for any $\alpha \in \mathbb{Q}$:

(i) $V^\alpha \tilde{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\tilde{M}$.

(ii) $tV^\alpha \tilde{M} \subset V^{\alpha+1}\tilde{M}$ and = holds for $\alpha \gg 0$.

(iii) $\partial_t V^\alpha \tilde{M} \subset V^{\alpha-1}\tilde{M}$.

(iv) $\partial_t - \alpha$ is nilpotent on $\text{Gr}^\alpha_V \tilde{M}$.

If it exists, it is unique.

3.5. Relation with the $b$-function. If $X$ is affine or Stein and relatively compact, then the multiplicity of a root $\alpha$ of $b_f(s)$ is given by the minimal polynomial of $s - \alpha$ on

$$
\text{Gr}_V^\alpha(\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}),
$$

using $\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f - t)$ with $s = -\partial_t t$.

Note that $V^\alpha \tilde{M}$ and $\mathcal{D}_X[s]f^{s+i}$ are ‘lattices’ of $\tilde{M}$, i.e.

$$
V^\alpha \tilde{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^\beta \tilde{M} \quad \text{for } \alpha \gg i \gg \beta,
$$

and $V^\alpha \tilde{M}$ is an analogue of the Deligne extension with eigenvalues in $[\alpha, \alpha+1]$. The existence of $V$ is equivalent to the existence of $b_f(s)$ locally.

3.6. Theorem (Kashiwara [21], [23], Malgrange [27]). The filtration $V$ exists on $\tilde{M} := i_{f+1}M$ for any holonomic $\mathcal{D}_X$-module $M$.

3.7. Remarks. (i) There are many ways to prove this theorem, since it is essentially equivalent to the existence of the $b$-function (in a generalized sense). One way is to use a resolution of singularities and reduce to the case where $CV(M)$ has normal crossings, if $M$ is regular.

(ii) The filtration $V$ is indexed by $\mathbb{Q}$ if $M$ is quasi-unipotent.

3.8. Relation with vanishing cycle functors. Let $\rho : X_t \rightarrow X_0$ be a ‘good’ retraction (using a resolution of singularities of $(X, X_0))$, where $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. Then we have canonical isomorphisms

$$
\psi_fC_X = R\rho_*C_{X_t}, \quad \varphi_fC_X = \psi_fC_X/C_{X_0},
$$

where $\psi_fC_X, \varphi_fC_X$ are nearby and vanishing cycle sheaves, see [13].

Let $F_x$ denote the Milnor fiber around $x \in X_0$. Then

$$
(H^j\psi_fC_X)_x = H^j(F_x, C), \quad (H^j\varphi_fC_X)_x = \widetilde{H}^j(F_x, C).
$$

For a $\mathcal{D}_X$-module $M$ admitting the V-filtration on $\tilde{M} = i_{f+1}M$, we define $\mathcal{D}_X$-modules

$$
\psi_fM = \bigoplus_{0 < \alpha \leq 1} \text{Gr}^\alpha_V \tilde{M}, \quad \varphi_fM = \bigoplus_{0 \leq \alpha < 1} \text{Gr}^\alpha_V \tilde{M}.
$$

3.9. Theorem (Kashiwara [23], Malgrange [27]). For a regular holonomic $\mathcal{D}_X$-module $M$, we have canonical isomorphisms

$$
\text{DR}_X \psi_f(M) = \psi_f \text{DR}_X(M)[-1],

\text{DR}_X \varphi_f(M) = \varphi_f \text{DR}_X(M)[-1],
$$
and $\exp(-2\pi i \partial_t t)$ on the left-hand side corresponds to the monodromy $T$ on the right-hand side.

3.10. Definition. Let

$$R_f = \{ \text{roots of } b_f(-s) \},$$

$$\alpha_f = \min R_f,$$

$m_\alpha$ : the multiplicity of $\alpha \in R_f.$

(Similarly for $R_{f,x}$, etc. for $b_{f,x}(s)$.)

3.11. Theorem (Kashiwara [20]). $R_f \subset \mathbb{Q}_{>0}.$

(This is proved by using a resolution of singularities.)

3.12. Theorem (Kashiwara [23], Malgrange [27]).

(i) $e^{-2\pi i R_f} = \{ \text{the eigenvalues of } T \text{ on } H^j(F_x, \mathbb{C}) \text{ for } x \in X_0, j \in \mathbb{Z} \},$

(ii) $m_\alpha \leq \min \{ i \mid N^i \psi_{f,\lambda} C_X = 0 \} \text{ with } \lambda = e^{-2\pi i \alpha},$

where $\psi_{f,\lambda} = \text{Ker}(T_{s-\lambda}) \subset \psi_f, N = \log T_u \text{ with } T = T_s T_u.$

(This is a corollary of the above Theorem (3.9) of Kashiwara and Malgrange.)

4. Relation with other invariants.

4.1. Microlocal $b$-function. We define $\tilde{R}_f, \tilde{m}_\alpha, \tilde{\alpha}_f$ with $b_f(s)$ replaced by the microlocal (or reduced) $b$-function

$$(4.1.1) \quad \tilde{b}_f(s) := b_f(s)/(s + 1).$$

This $\tilde{b}_f(s)$ coincides with the monic polynomial of the least degree satisfying

$$(4.1.2) \quad \tilde{b}_f(s) \delta(f - t) = \tilde{P} \partial_t^{-1} \delta(f - t) \text{ with } \tilde{P} \in \mathcal{D}_X[s, \partial_t^{-1}].$$

Put $n = \dim X$. Then

4.2. Theorem. $\tilde{R}_f \subset [\tilde{\alpha}_f, n - \tilde{\alpha}_f], \quad \tilde{m}_\alpha \leq n - \tilde{\alpha}_f - \alpha + 1.$

(The proof uses the filtered duality for $\varphi_f$, see [35].)

4.3. Spectrum. We define the spectrum by $\text{Sp}(f, x) = \sum_\alpha n_\alpha t^\alpha$ with

$$(4.3.1) \quad n_\alpha := \sum_j (-1)^{j-n+1} \dim \text{Gr}_F \tilde{H}^j(F_x, \mathbb{C}),$$

where $p = [n - \alpha], \lambda = e^{-2\pi i \alpha}$, and $F$ is the Hodge filtration (see [12]) of the mixed Hodge structure on the Milnor cohomology, see [44]. We define

$$(4.3.2) \quad E_f = \{ \alpha \mid n_\alpha \neq 0 \} \text{ (called the exponents).}$$

4.4. Remarks. (i) If $f$ has an isolated singularity at the origin, then $\tilde{\alpha}_{f,x}$ coincides with the minimal exponent as a corollary of results of Malgrange [26], Varchenko [45], Scherk-Steenbrink [41].

(ii) If $f$ is weighted-homogeneous with an isolated singularity at the origin, then by Kashiwara (unpublished)

$$(4.4.1) \quad \tilde{R}_f = E_f, \quad \max \tilde{R}_f = n - \tilde{\alpha}_f, \quad \tilde{m}_\alpha = 1 (\alpha \in \tilde{R}_f).$$
If \( f = \sum_i x_i^2 \), then \( \tilde{\alpha}_f = n/2 \) and this follows from the above Theorem (4.2). By Steenbrink [42], we have moreover

\[
\text{Sp}(f, x) = \prod_i (t - t_{w_i})/(t_{w_i} - 1),
\]
where \( (w_1, \ldots, w_n) \) is the weights of \( f \), i.e. \( f \) is a linear combination of monomials \( x_1^{m_1} \cdots x_n^{m_n} \) with \( \sum_i w_i m_i = 1 \).

### 4.5. Malgrange’s formula (isolated singularities case)

We have the Brieskorn lattice [5] and its saturation defined by

\[
H''_f = \Omega^n_{X,x}/df \wedge d\Omega^{n-2}_{X,x}, \quad \tilde{H}_f'' = \sum_{i \geq 0} (t \partial_t)^i H''_f \subset H''_f[t^{-1}].
\]
These are finite \( \mathbb{C}\{t\} \)-modules with a regular singular connection.

### 4.6. Theorem (Malgrange [26])

The reduced \( b \)-function \( \tilde{b}_f(s) \) coincides with the minimal polynomial of \(-\partial_t\) on \( \tilde{H}_f''/t \tilde{H}_f'' \).

(The above formula of Kashiwara on \( b \)-function (4.4.1) can be proved by using this together with Brieskorn’s calculation.)

### 4.7. Asymptotic Hodge structure (Varchenko [45], Scherk-Steenbrink [41])

In the isolated singularity case we have

\[
F^p H^{n-1}(F, \mathbb{C})_\lambda = \text{Gr}_V^\alpha \tilde{H}_f''[t^{-1}],
\]
using the canonical isomorphism (4.7.2), where \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha} \), and \( V \) on \( H''_f[t^{-1}] \) is the filtration of Kashiwara and Malgrange.

(This can be generalized to the non-isolated singularity case using mixed Hodge modules.)

### 4.8. Reformulation of Malgrange’s formula

We define

\[
\tilde{F}^p H^{n-1}(F, \mathbb{C})_\lambda = \text{Gr}_V^\alpha \tilde{H}_f''[t^{-1}],
\]
using the canonical isomorphism (4.7.2), where \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha} \). Then

\[
\tilde{m}_\alpha = \text{the minimal polynomial of } N \text{ on } \text{Gr}_F^p H^{n-1}(F, \mathbb{C})_\lambda.
\]

### 4.9. Remark

If \( f \) is weighted homogeneous with an isolated singularity, then

\[
\tilde{F} = F, \quad \tilde{R}_f = E_f \quad \text{(by Kashiwara)}.
\]

If \( f \) is not weighted homogeneous (but with isolated singularities), then

\[
\tilde{R}_f \subset \bigcup_{k \in \mathbb{N}} (E_f - k), \quad \tilde{\alpha}_f = \min \tilde{R}_f = \min E_f.
\]

### 4.10. Example

If \( f = x^5 + y^4 + x^3 y^2 \), then

\[
E_f = \left\{ \frac{i}{5} + \frac{j}{4} : 1 \leq i \leq 4, 1 \leq j \leq 3 \right\}, \quad \tilde{R}_f = E_f \cup \left\{ \frac{11}{20} \right\} \setminus \left\{ \frac{31}{20} \right\}.
\]
More generally, if \( f = g + h \) with \( g \) weighted homogeneous and \( h \) is a linear combination of monomials of higher degrees, then \( E_f = E_g \) but \( \tilde{R}_f \neq \tilde{R}_g \) if \( f \) is a non trivial deformation.

4.11. Relation with rational singularities [34]. Assume \( D := f^{-1}(0) \) is reduced. Then \( D \) has rational singularities if and only if \( \tilde{\alpha}_f > 1 \). Moreover, \( \omega_D/\rho_*\omega_{\tilde{D}} \simeq F_{1-n\varphi f}O_X \), where \( \rho : \tilde{D} \to D \) is a resolution of singularities.

In the isolated singularities case, this was proved in 1981 (see [31]) using the coincidence of \( \tilde{\alpha}_f \) and the minimal exponent.

4.12. Relation with the pole order filtration [34]. Let \( P \) be the pole order filtration on \( O_X(*D) \), i.e. \( P_i = O_X((i + 1)D) \) if \( i \geq 0 \), and \( P_i = 0 \) if \( i < 0 \). Let \( F \) be the Hodge filtration on \( O_X(*D) \). Then \( F_i \subset P_i \) in general, and \( F_i = P_i \) on a neighborhood of \( x \) for \( i \leq \tilde{\alpha}_{f,x} - 1 \).

(For the proof we need the theory of microlocal \( b \)-functions [35].)

4.13. Remark. In case \( X = \mathbb{P}^n \), replacing \( \tilde{\alpha}_{f,x} \) with \( [(n - r)/d] \) where \( r = \dim \text{Sing } D \) and \( d = \deg D \), the assertion was obtained by Deligne (unpublished).

5. Relation with multiplier ideals.

5.1. Multiplier ideals. Let \( D = f^{-1}(0) \), and \( J(X,\alpha D) \) be the multiplier ideals for \( \alpha \in \mathbb{Q} \), i.e.

\[
J(X, \alpha D) = \rho_*\omega_{\tilde{X}/X}(-\sum_i [\alpha m_i \tilde{D}_i]),
\]

where \( \rho : (\tilde{X}, \tilde{D}) \to (X, D) \) is an embedded resolution and \( \tilde{D} = \sum_i m_i \tilde{D}_i := \rho^* D \).

There exist jumping numbers \( 0 < \alpha_0 < \alpha_1 < \cdots \) such that

\[
J(X, \alpha_j D) = J(X, \alpha D) \neq J(X, \alpha_{j+1} D) \quad \text{for} \quad \alpha_j \leq \alpha < \alpha_{j+1}.
\]

Let \( V \) denote also the induced filtration on

\[ O_X \subset O_X[\partial_t]\delta(f - t). \]

5.2. Theorem (Budur, S. [10]). If \( \alpha \) is not a jumping number,

\[
J(X, \alpha D) = V^\alpha O_X.
\]

For \( \alpha \) general we have for \( 0 < \varepsilon \ll 1 \)

\[
J(X, \alpha D) = V^{\alpha + \varepsilon} O_X, \quad V^\alpha O_X = J(X, (\alpha - \varepsilon) D).
\]

Note that \( V \) is left-continuous and \( J(X, \alpha D) \) is right-continuous, i.e.

\[
V^\alpha O_X = V^{\alpha - \varepsilon} O_X, \quad J(X, \alpha D) = J(X, (\alpha + \varepsilon) D).
\]

The proof of (5.2) uses the theory of bifiltered direct images [32], [33] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [16], and of Lichtin, Yano and Kollár [25]:
6.4. Theorem (Budur, Mustată, S. [8]).

\( c \) of the ideal of \( Z \) of \( t \)

\[ \prod_{i} \text{well-definedness does not hold without the term } (6) \]

6.3. Equivalent definition. The \( \alpha_{f} = \text{minimal jumping number} \), see [25].

Define \( \alpha'_{f,x} = \min_{y \neq x} \{ \alpha_{f,y} \} \). Then

\[ (\text{This does not hold without the assumption on } \xi \text{ nor for } (\alpha'_{f,x}, 1).) \]

6.1. Let \( Z \) be a closed subvariety of a smooth \( X \), and \( f = (f_1, \ldots, f_r) \) be generators of the ideal of \( Z \) (which is not necessarily reduced nor irreducible). Define the action of \( t_j \) on

\[ \mathcal{O}_{X} \left[ \frac{1}{f_1 \cdots f_r} [s_1, \ldots, s_r] \prod_i f_i^{s_i} \right], \]

by \( t_j(s_i) = s_i + 1 \) if \( i = j \), and \( t_j(s_i) = s_i \) otherwise. Put \( s_{i,j} := st_i^{-1}t_j, s = \sum_i s_i \). Then \( b_f(s) \) is the monic polynomial of the least degree satisfying

\[ (6.1.1) \quad b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^r P_k t_k \prod_i f_i^{s_i}, \]

where \( P_k \) belong to the ring generated by \( \mathcal{D}_X \) and \( s_{i,j} \).

Here we can replace \( \prod_i f_i^{s_i} \) with \( \prod_i \delta(t_i - f_i) \), using the direct image by the graph of \( f : X \to C^r \). Then the existence of \( b_f(s) \) follows from the theory of the \( V \)-filtration of Kashiwara and Malgrange. This \( b \)-function has appeared in work of Sabbah [30] and Gyoja [18] for the study of \( b \)-functions of several variables.

6.2. Theorem (Budur, Mustată, S. [8]). Let \( c = \text{codim}_X Z \). Then \( b_Z(s) := b_f(s - c) \)
depends only on \( Z \) and is independent of the choice of \( f = (f_1, \ldots, f_r) \) and also of \( r \).

6.3. Equivalent definition. The \( b \)-function \( b_f(s) \) coincides with the monic polynomial of the least degree satisfying

\[ (6.3.1) \quad b_f(s) \prod_i f_i^{s_i} \in \sum_{|c|=1} D_X[s] \prod_{\alpha < 0} (\alpha \cdot c_i) \prod_i f_i^{s_i + c_i}, \]

where \( c = (c_1, \ldots, c_r) \in \mathbb{Z}^r \) with \( |c| := \sum_i c_i = 1 \). Here \( D_X[s] = D_X[s_1, \ldots, s_r] \).

This is due to Mustăță, and is used in the monomial ideal case. Note that the well-definedness does not hold without the term \( \prod_{\alpha < 0} (\alpha \cdot c_i) \).

We have the induced filtration \( V \) by

\[ \mathcal{O}_X \subset i_{f,x} \mathcal{O}_X = \mathcal{O}_X[\partial_1, \ldots, \partial_r] \prod t_i \delta(t_i - f_i). \]

6.4. Theorem (Budur, Mustăță, S. [8]). If \( \alpha \) is not a jumping number,

\[ (6.4.1) \quad J(X, \alpha Z) = V^\alpha \mathcal{O}_X. \]
For $\alpha$ general we have for $0 < \varepsilon \ll 1$

\[
(6.4.2) \quad \mathcal{J}(X, \alpha Z) = V^{\alpha + \varepsilon} \mathcal{O}_X, \quad V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z).
\]

6.5. Corollary (Budur, Mustaţă, S. [8]). We have the inclusion

\[
(6.5.1) \quad \{\text{Jumping numbers}\} \cap [\alpha_f, \alpha_f + 1) \subset R_f.
\]

6.6. Theorem (Budur, Mustaţă, S. [8]). If $Z$ is reduced and is a local complete intersection, then $Z$ has only rational singularities if and only if $\alpha_f = r$ with multiplicity $1$.

7. Monomial ideal case.

7.1. Definition. Let $a \subset \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ a monomial ideal. We have the associated semigroup defined by

\[
\Gamma_a = \{u \in \mathbb{N}^n \mid x^u \in a\}.
\]

Let $P_a$ be the convex hull of $\Gamma_a$ in $\mathbb{R}^n_{\geq 0}$. For a face $Q$ of $P_a$, define

\[
M_Q : \text{the subsemigroup of } \mathbb{Z}^n \text{ generated by } u - v \text{ with } u \in \Gamma_a, v \in \Gamma_a \cap Q.
\]

\[
M'_Q = v_0 + M_Q \text{ for } v_0 \in \Gamma_a \cap Q \text{ (this is independent of } v_0).\]

For a face $Q$ of $P_a$ not contained in any coordinate hyperplane, take a linear function with rational coefficients $L_Q : \mathbb{R}^n \to \mathbb{R}$ whose restriction to $Q$ is 1. Let

\[
V_Q : \text{the linear subspace generated by } Q.
\]

\[
e = (1, \ldots, 1).
\]

\[
R_Q = \{L_Q(u) \mid u \in (e + (M_Q \setminus M'_Q)) \cap V_Q\},
\]

\[
R_a = \{\text{roots of } b_a(-s)\}.
\]

7.2. Theorem (Budur, Mustaţă, S. [9]). We have $R_a = \bigcup_Q R_Q$ with $Q$ faces of $P_a$ not contained in any coordinate hyperplanes.

Outline of the proof. Let $f_j = \prod_i x_i^{a_{i,j}}$, $\ell_i(s) = \sum_j a_{i,j}s_j$. Define

\[
g_c(s) = \prod_{c_i < 0} \left(\frac{s_i}{-c_i}\right) \prod_{\ell_i(c) > 0} \left(\frac{\ell_i(s) + \ell_i(c)}{\ell_i(c)}\right).
\]

Let $I_a \subset \mathbb{C}[s]$ be the ideal generated by $g_c(s)$ with $c \in \mathbb{Z}^r, \sum_i c_i = 1$. Then

7.3. Proposition (Mustaţă). The $b$-function $b_a[s]$ of the monomial ideal $a$ is the monic generator of $\mathbb{C}[s] \cap I_a$, where $s = \sum_i s_i$.

Using this, Theorem (7.2) follows from elementary computations.

7.4. Case $n = 2$. Here it is enough to consider only 1-dimensional $Q$ by (7.2). Let $Q$ be a compact face of $P_a$ with $\{v^{(1)}, v^{(2)}\} = \partial Q$, where $v^{(i)} = (v_1^{(i)}, v_2^{(i)})$ with $v_1^{(1)} < v_1^{(2)}$, $v_2^{(1)} > v_2^{(2)}$. Let

\[
G : \text{the subgroup generated by } u - v \text{ with } u, v \in Q \cap \Gamma_a.
\]

$v^{(3)} \in Q \cap \mathbb{N}^2$ such that $v^{(3)} - v^{(1)}$ generates $G$.

\[
S_Q = \{(i, j) \in \mathbb{N}^2 \mid i < v_1^{(3)}, j > v_2^{(1)}\}.
\]
$S_Q^{[1]} = S \cap M_Q$, $S_Q^{[0]} = S_Q \setminus S_Q^{[1]}$.

Then

$$R_Q = \{L_Q(u + e) - k \mid u \in S_Q^{[k]} (k = 0, 1)\}.$$  

In the case $Q \subset \{x = m\}$, we have $R_Q = \{i/m \mid i = 1, \ldots, m\}$.

**7.5. Examples.** (i) If $a = (x^ay, xy^b)$, with $a, b \geq 2$, then

$$R_a = \{\frac{(b-1)i + (a-1)j}{ab-1} \mid 1 \leq i \leq a, 1 \leq j \leq b\}.$$  

(ii) If $a = (xy^5, x^3y^2, x^5y)$, then $S_Q^{[1]} = \emptyset$ and

$$R_a = \left\{\frac{5}{13}, \frac{i}{13} (7 \leq i \leq 17), \frac{19}{13}, \frac{j}{6} (3 \leq j \leq 9)\right\}.$$  

(iii) If $a = (xy^5, x^3y^2, x^4y)$, then $S_Q^{[1]} = \{(2, 4)\}$ for $\partial Q = \{(1, 5), (3, 2)\}$ with $L_Q(v_1, v_2) = (3v_1 + 2v_2)/13$, and

$$R_a = \left\{\frac{i}{13} (5 \leq i \leq 17), \frac{j}{5} (2 \leq j \leq 6)\right\}.$$  

Here $19/13$ is shifted to $6/13$.

**7.6. Comparison with exponents.** If $n = 2$ and $f$ has a nondegenerate Newton polygon with compact faces $Q$, then by Steenbrink [43]

$$E_f \cap (0, 1] = \bigcup_Q E_f^{\leq 1}$$

with $E_f^{\leq 1} = \{L_Q(u) \mid u \in \{0\} \cup Q \cap \mathbb{Z}_{\geq 0}\}$,

where $\{0\} \cup Q$ is the convex hull of $\{0\} \cup Q$. Here we have the symmetry of $E_f$ with center 1.

**7.7. Another comparison.** If $a = (x_1^{a_1}, \ldots, x_n^{a_n})$, then

$$R_a = \{\sum p_i/a_i \mid 1 \leq p_i \leq a_i\}.$$  

On the other hand, if $f = \sum_i x_i^{a_i}$, then

$$\tilde{R}_f = E_f = \{\sum p_i/a_i \mid 1 \leq p_i \leq a_i - 1\}.$$  

**8. Hyperplane arrangements.**

**8.1.** Let $D$ be a central hyperplane arrangement in $X = \mathbb{C}^n$. Here, central means an affine cone of $Z \subset \mathbb{P}^{n-1}$. Let $f$ be the reduced equation of $D$ and $d := \deg f > n$. Assume $D$ is not the pull-back of $D' \subset \mathbb{C}^{n'} (n' < n)$.

**8.2. Theorem.** (i) $\max R_f < 2 - \frac{1}{d}$. (ii) $m_1 = n$.

Proof of (i) uses a partial generalization of a solution of Aomoto’s conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([17], [40]) together with a generalization of Malgrange’s formula (4.8) as below:
8.3. Theorem (Generalization of Malgrange’s formula) [36]. There exists a pole order filtration $P$ on $H^{n-1}(F_0, C)_{\lambda}$ such that if $(\alpha + N) \cap R'_f = \emptyset$, then

$$\alpha \in R_f \iff \text{Gr}^P_{\lambda} H^{n-1}(F_0, C)_{\lambda} \neq 0,$$

with $p = [n - \alpha]$, $\lambda = e^{-2\pi i \alpha}$, where $R'_f = \bigcup_{x \notin 0} R_{f,x}$.

This reduces the proof of (8.2)(i) to

$$P^i H^{n-1}(F_0, C)_{\lambda} = H^{n-1}(F_0, C)_{\lambda},$$

for $i = n - 1$ if $\lambda = 1$ or $e^{2\pi i / d}$, and $i = n - 2$ otherwise.

8.4. Construction of the pole order filtration $P$. Let $U = \mathbb{P}^{n-1} \setminus Z$, and $F_0 = f^{-1}(0) \subset C^n$. Then $F_0 = \tilde{U}$ with $\pi : \tilde{U} \to U$ a $d$-fold covering ramified over $Z$. Let $L^{(k)}$ be the local systems of rank 1 on $U$ such that $\pi_* C = \bigoplus_{0 \leq i < d} L^{(k)}$ and $T$ acts on $L^{(k)}$ by $e^{-2\pi ik/d}$. Then

$$H^1(U, L^{(k)}) = H^1(F_0, C)_{\lambda(e^{-k/d})},$$

and $P$ is induced by the pole order filtration on the meromorphic extension $L^{(k)}$ of $L^{(k)} \otimes_{\mathcal{O}_U} \mathcal{O}_U$ over $\mathbb{P}^{n-1}$, see [15], [36], [37]. This is closely related to:

8.5. Solution of Aomoto’s conjecture ([17], [40]). Let $Z_i$ be the irreducible components of $Z$ $(1 \leq i \leq d)$, $g_i$ be the defining equation of $Z_i$ on $\mathbb{P}^{n-1} \setminus Z_d$ $(i < d)$, and $\omega := \sum_{i < d} \alpha_i \omega_i$ with $\omega_i = dg_i / g_i$, $\alpha_i \in C$. Let $\nabla$ be the connection on $\mathcal{O}_U$ such that $\nabla u = du + \omega \wedge u$. Set $\alpha_d = -\sum_{i < d} \alpha_i$. Then $H^*_{\text{DR}}(U, (\mathcal{O}_U, \nabla))$ is calculated by

$$(A^*_\alpha, \omega \wedge) \text{ with } A^*_\alpha = \sum C\omega_1 \wedge \cdots \wedge \omega_{i_p},$$

if $\sum_{Z_i \supset L} \alpha_i \notin N \setminus \{0\}$ for any dense edge $L \subset Z$ (see (8.7) below). Here an edge is an intersection of $Z_i$.

For the proof of (8.2)(ii) we have

8.6. Proposition. $N^{n-1} \psi_{f,\lambda} C \neq 0$ if $\text{Gr}^W_{2n-2} H^{n-1}(F_x, C)_{\lambda} \neq 0$.

(Indeed, $N^{n-1} : \text{Gr}^W_{2n-2} \psi_{f,\lambda} C \rightleftharpoons \text{Gr}^0_{\psi_{f,\lambda}} C$ by the definition of $W$, and the assumption of (8.6) implies $\text{Gr}^W_{2n-2} \psi_{f,\lambda} C \neq 0$.)

Then we get (8.2)(ii), since $\omega_1 \wedge \cdots \wedge \omega_{i_{n-1}} \neq 0$ in $\text{Gr}^W_{2n-2} H^{n-1}(\mathbb{P}^{n-1} \setminus Z, C) = \text{Gr}^W_{2n-2} H^{n-1}(F_x, C)_{\lambda}$.

8.7. Dense edges. Let $D = \bigcup_i D_i$ be the irreducible decomposition. Then $L = \bigcap_{i \in I} D_i$ is called an edge of $D$ ($I \neq \emptyset$).

We say that an edge $L$ is dense if $\{D_i / L | D_i \supset L\}$ is indecomposable. Here $C^n \supset D$ is called decomposable if $C^n = C^{n'} \times C^{n''}$ such that $D$ is the union of the pull-backs from $C^{n'}$, $C^{n''}$ with $n', n'' \neq 0$.

Set $m_L = \#\{D_i | D_i \supset L\}$. For $\lambda \in C$, define

$$\mathcal{D}E(D) = \{\text{dense edges of } D\}, \quad \mathcal{D}E(D, \lambda) = \{L \in \mathcal{D}E(D) | \lambda^{m_L} = 1\}.$$
We say that \( L, L' \) are strongly adjacent if \( L \subset L' \) or \( L \supset L' \) or \( L \cap L' \) is non-dense. Let
\[
m(\lambda) = \max \{|S| \mid S \subset \mathcal{DE}(D, \lambda) \text{ such that } \forall L, L' \in S \text{ are strongly adjacent}\}.
\]

8.8. Theorem [37]. \( m_\alpha \leq n(\lambda) \) with \( \lambda = e^{-2\pi i \alpha} \).

8.9. Corollary. \( R_\alpha \subset \bigcup_{L \in \mathcal{DE}(D)} Zm^*_L \).

8.10. Corollary. If \( \gcd(m_L, m_{L'}) = 1 \) for any strongly adjacent \( L, L' \in \mathcal{DE}(D) \), then \( m_\alpha = 1 \) for any \( \alpha \in R_\alpha \setminus \mathbb{Z} \).

Theorem 2 follows from the canonical resolution of singularities \( \pi : (\tilde{X}, \tilde{D}) \to (\mathbb{P}^{n-1}, D) \) due to [40], which is obtained by blowing up along the proper transforms of the dense edges. Indeed, \( \mult(\tilde{D}(\lambda)_{\text{red}}) \leq m(\lambda) \), where \( \tilde{D}(\lambda) \) is the union of \( \tilde{D}_i \) such that \( \lambda^{\tilde{m}_i} = 1 \) and \( \tilde{m}_i = \mult_{\tilde{D}_i} \tilde{D} \).

8.11. Theorem (Mustață [29]). For a central arrangement,
\[(8.11.1) \quad J(X, \alpha D) = I_0^k \text{ with } k = [d\alpha] - n + 1 \text{ if } \alpha < \alpha_f,\]
where \( I_0 \) is the ideal of 0 and \( \alpha_f = \min_{x \neq 0} \{\alpha_{f,x}\} \).

(This holds for the affine cone of any divisor on \( \mathbb{P}^{n-1} \), see [36].)

8.12. Corollary. We have \( \dim F^{n-1}H^{n-1}(F_0, C)_{e(-k/d)} = \binom{k-1}{n-1} \) for \( 0 < k/d < \alpha_f' \), and the same holds with \( F \) replaced by \( P \).

8.13. Corollary. \( \alpha_f = \min(\alpha_f', \frac{n}{d}) < 1. \)

(Note that \( \alpha_f \) coincides with the minimal jumping number.)

8.14. Generic case. If \( D \) is a generic central hyperplane arrangement, then
\[(8.14.1) \quad b_f(s) = (s + 1)^{n-1} \prod_{j=0}^{2d-2} (s + \frac{j}{d})\]
by U. Walther [46] (except for the multiplicity of \(-1\)). He uses a completely different method.

Note that Theorems (8.2) and (8.8) imply that the left-hand side divides the right-hand side of (8.14.1), and the equality follows using also (8.12).

8.15. Explicit calculation. Let \( \alpha = k/d, \lambda = e^{-2\pi i \alpha} \) for \( k \in \{1, \ldots, d\} \). If \( \alpha \geq \alpha_f' \), we assume there is \( I \subset \{1, \ldots, d-1\} \) such that \( |I| = k - 1 \), and the condition of [40]
\[(8.15.1) \quad \sum_{Z_i \supset L} \alpha_i \notin \mathbb{N} \setminus \{0\} \text{ for any dense edge } L \subset Z, \]
is satisfied for
\[(8.15.2) \quad \alpha_i = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise.}\]

Let \( V(I) \) be the subspace of \( H^{n-1}A^*_\alpha \) generated by
\[\omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-1}} \text{ for } \{i_1, \ldots, i_{n-1}\} \subset I.\]

8.16. Theorem. Let \( \alpha = k/d, \lambda = e^{-2\pi i \alpha} \) for \( k \in \{1, \ldots, d\} \). Then
8.17. Theorem [37]. Assume $n = 3$, $\text{mult}_Z \leq 3$ for any $z \in Z \subset P^2$, and $d \leq 7$. Let $\nu_3$ be the number of triple points of $Z$, and assume $\nu_3 \neq 0$. Then

\begin{equation}
(8.17.1) \quad b_f(s) = (s + 1) \prod_{i=2}^{n} (s + \frac{i}{3}) \prod_{j=3}^{n} (s + \frac{j}{d}),
\end{equation}

with $r = 2d - 2$ or $2d - 3$. We have $r = 2d - 2$ if $\nu_3 < d - 3$, and the converse holds for $d < 7$. In case $d = 7$, we have $r = 2d - 3$ for $\nu_3 > 4$, however, for $\nu_3 = 4$, $r$ can be both $2d - 2$ and $2d - 3$.

8.18. Remarks. (i) We have $\nu_3 < d - 3$ if and only if

\begin{equation}
(8.18.1) \quad \chi(U) = \frac{(d - 2)(d - 3)}{2} - \nu_3 > \frac{(d - 3)(d - 4)}{2} = \binom{d - 3}{2}.
\end{equation}

(ii) By (8.4.1) we have $\chi(U) = h^2(F_0, C)\lambda - h^1(F_0, C)\lambda$ if $\lambda^d = 1$ and $\lambda \neq 0$.

(iii) Let $\nu_i'$ be the number of $i$-ple points of $Z' := Z \cap C^2$. Then by [6]

\begin{equation}
(8.18.2) \quad b_0(U) = 1, \quad b_1(U) = d - 1, \quad b_2(U) = \nu'_2 + 2\nu'_3,
\end{equation}

8.19. Examples. (i) For $(x^2 - 1)(y^2 - 1) = 0$ in $C^2$ with $d = 5$, (8.17.1) holds with $r = 7$, and $8/5 \notin R_f$. In this case we do not need to take $I$, because $(d - 2)/d = 3/5 < \alpha' = 2/3$. We have $b_1(U) = b_2(U) = 4$ and $h^2(F_0, C)\lambda = \chi(U) = 1$ if $\lambda^5 = 1$ and $\lambda \neq 1$. So $j/5 \in R_f$ for $3 \leq j \leq 7$ by (a), (b), (c), and $8/5 \notin R_f$ by (d).

(ii) For $(x^2 - 1)(y^2 - 1)(x + y) = 0$ in $C^2$ with $d = 6$, (8.17.1) holds with $r = 9$, and $10/6 \notin R_f$. In this case we have $b_1(U) = 5, b_2(U) = 6, \chi(U) = 2, h^1(F_0, C)\lambda = 1, h^2(F_0, C)\lambda = 3$ for $\lambda = e^{\pm 2\pi i/3}$. Then $4/6 \in R_f$ by (e) and $10/6 \notin R_f$ by (f), where $I^c$ corresponds to $(x + 1)(y + 1) = 0$. For other $j/6$, the argument is the same as in (i).

(iii) For $(x^2 - y^2)(x^2 - 1)(y + 2) = 0$ in $C^2$ with $d = 6$, (8.17.1) holds with $r = 10$, and $10/6 \in R_f$. In this case we have $b_1(U) = 5, b_2(U) = 9, \chi(U) = 5, h^1(F_0, C)\lambda = 0, h^2(F_0, C)\lambda = 5$ for $\lambda = e^{\pm 2\pi i/3}$. Then $4/6 \in R_f$ by (e) and $10/6 \in R_f$ by (c), where $I^c$ corresponds to $(x + 1)(y + 2) = 0$.

(iv) For $(x^2 - y^2)(x^2 - 1)(y^2 - 1) = 0$ in $C^2$ with $d = 7$, (8.17.1) holds with $r = 11$, and $12/7 \notin R_f$. In this case we have $b_1(U) = 6, b_2(U) = 9, \chi(U) = 4, h^2(F_0, C)\lambda = 4$ if $\lambda^7 = 1$ and $\lambda \neq 1$. Then $5/7 \in R_f$ by (e) and $12/7 \notin R_f$ by (f), where $I^c$ corresponds to $(x + 1)(y + 1) = 0$. Note that $5/7$ is not a jumping number.
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