

FINITE GENERATION OF TORIC FGA ALGEBRAS

FLORIN AMBRO

FGA algebras were recently introduced by Shokurov [4], who showed that their finite generation in dimension d implies the existence of flips in dimension $d + 1$. Shokurov showed that FGA algebras are finitely generated in dimension 1 and 2, and he conjectured this to be true in any dimension. We answer this conjecture in the affirmative, in the toric case [1].

FGA algebras. Let $(X/S, B)$ be the data consisting of a proper surjective morphism $\pi: X \rightarrow S$ of complex algebraic varieties, and a log variety (X, B) with Kawamata log terminal singularities such that $-(K + B)$ is nef and big relative to π .

The simplest example of an FGA algebra is the graded \mathcal{O}_S -algebra

$$\mathcal{R}_{X/S}(D) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD),$$

for a Cartier divisor D on X . The finite generation of $\mathcal{R}_{X/S}(D)$ is a consequence of the Log Minimal Model Program in dimension $\dim(X)$.

In general, an FGA algebra is a graded subalgebra $\mathcal{L} \subseteq \mathcal{R}_X(D)$ satisfying an extra property called *asymptotic saturation*. To explain this, we need some preparation. The normalization of \mathcal{L} is also a graded subalgebra of $\mathcal{R}_{X/S}(D)$, and it has a presentation

$$\bar{\mathcal{L}} = \bigoplus_{i=0}^{\infty} \pi_{i*} \mathcal{O}_{X_i}(M_i),$$

where $\mu_i: X_i \rightarrow X$ is a birational modification, $\pi_i = \pi \circ \mu_i$ and M_i is a π_i -free divisor on X_i . We may replace the X_i 's by higher birational models, so that X_i is nonsingular and contains a simple normal crossings divisor which supports both $K_{X_i} - \mu_i^*(K + B)$ and M_i . Then \mathcal{L} is called asymptotically saturated if

$$\pi_{i*} \mathcal{O}_{X_i}([K_{X_i} - \mu_i^*(K + B) + \frac{j}{i} M_i]) \subseteq \pi_{j*} \mathcal{O}_{X_j}(M_j), \forall i, j \geq 1.$$

Asymptotic saturation involves a priori infinitely many birational models of X . If $\dim(X) \leq 2$, its particular case $i = j$ bounds the singularities on X of the (non-complete) linear systems $|\mathcal{L}_i| \subset |iD|$, and this

in turn can be used to restate asymptotic saturation in terms of just one birational model of X . Finite generation follows then by a standard argument. It is an interesting open question whether a similar approach works for $\dim(X) \geq 3$.

In the toric case, we do not know if the above approach works. Instead, we can interpret toric asymptotic saturation only in terms of the limit $\lim_{i \rightarrow \infty} \frac{1}{i} M_i$, which can be regarded as (the support function of) a convex set \square . The finite generation of toric FGA algebras becomes then a criterion for \square to be rational and polyhedral.

Convex sets. Let N be a lattice, with dual lattice M , let $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ be the scalar extensions, with induced pairing $\langle \cdot, \cdot \rangle: M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$. For a function $h: N_{\mathbb{R}} \rightarrow \mathbb{R}$, define

$$\square_h = \{m \in M_{\mathbb{R}}; \langle m, e \rangle \geq h(e), \forall e \in N_{\mathbb{R}}\}$$

$$\overset{\circ}{\square}_h = \{m \in M_{\mathbb{R}}; \langle m, e \rangle > h(e), \forall e \in N_{\mathbb{R}} \setminus \{0\}\}.$$

For a non-empty compact convex set $\square \subset M_{\mathbb{R}}$, its support function is

$$h_{\square}: N_{\mathbb{R}} \rightarrow \mathbb{R}, h_{\square}(e) = \min_{m \in \square} \langle m, e \rangle.$$

The set of non-empty compact convex subsets of $M_{\mathbb{R}}$ is in bijection with the set of functions $h: N_{\mathbb{R}} \rightarrow \mathbb{R}$ satisfying

- (i) *positively homogeneous*: $h(\lambda e) = \lambda h(e)$ for $\lambda \geq 0, e \in N_{\mathbb{R}}$.
- (ii) *upper convex*: $h(e_1 + e_2) \geq h(e_1) + h(e_2)$ for $e_1, e_2 \in N_{\mathbb{R}}$.

The correspondence is given by $\square \mapsto h_{\square}$ and $h \mapsto \square_h$.

Definition 1. A *log discrepancy function* is a positively homogeneous, continuous function $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $\psi(e) > 0$ for $e \neq 0$, and $\{e \in N_{\mathbb{R}}; \psi(e) \leq 1\}$ is a compact set.

Toric FGA algebras. For simplicity, we assume that S is a point. Thus $X = T_N \text{emb}(\Delta)$ is a proper torus embedding, $B = \sum_{e \in \Delta(1)} b_e V(e)$ is an invariant \mathbb{Q} -divisor and $K + B$ is \mathbb{Q} -Cartier. This means that there exists a function $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $\psi(e) = 1 - b_e$ for every $e \in \Delta(1)$, and ψ is Δ -linear. Since (X, B) has Kawamata log terminal singularities, ψ is a log discrepancy function. The terminology is inspired by the following property: a primitive lattice point $e \in N$ defines a toric valuation v_e of X , and the log discrepancy of (X, B) at v_e is exactly $\psi(e)$. Also, note that $-(K + B)$ is nef if and only if $-\psi$ is upper convex.

Any invariant normal algebra is of the form

$$\mathcal{L} = \bigoplus_{i=0}^{\infty} \left(\bigoplus_{m \in M \cap \square_i} \mathbb{C} \chi^m \right)$$

where $(\square_i)_{i \geq 0}$ is a sequence of lattice convex polytopes in $M_{\mathbb{R}}$ satisfying the following properties:

- $\square_0 = \{0\}$.
- $\square_i + \square_j \subset \square_{i+j}$ for $i, j \geq 1$.
- $\square := \bigcup_{i \geq 1} \frac{1}{i} \square_i$ is a bounded convex set.

The algebra is finitely generated if and only if $\square_i = \square$ for some i . For an example of such a sequence, fix a compact convex set $\square \subset M_{\mathbb{R}}$, and let \square_i be the convex hull of $M \cap i\square$.

Since limits of upper convex functions converge uniformly on compact sets, the asymptotic saturation of \mathcal{L} with respect to (X, B) is equivalent to the following Diophantine property

$$M \cap \overset{\circ}{\square}_{jh-\psi} \subset \square_j, \forall j \geq 1.$$

Then \mathcal{L} is finitely generated if and only if \square is a rational polytope, i.e. the convex hull of finitely many rational points, and this follows from the following result:

Theorem 2. *Let ψ be a log discrepancy function and let $\square \subset M_{\mathbb{R}}$ be a non-empty compact convex set with support function h . Assume that the following properties hold:*

- (i) $M \cap \overset{\circ}{\square}_{jh-\psi} \subset j\square$ for every $j \geq 1$.
- (ii) $rh - \psi$ is upper convex, for some $r > 0$.

Then \square is a rational polytope.

Theorem 2 is proved by induction on dimension. Consider the unit sphere $S(N_{\mathbb{R}})$ with respect to some norm on $N_{\mathbb{R}}$. Using Diophantine Approximation [2], we show that every point $e \in S(N_{\mathbb{R}})$ is contained in the relative interior of a rational polyhedral cone σ_e on which h_{\square} is rational and linear with respect to some fan with support σ_e . If $\dim(\sigma_e) = \dim(N)$ for every e , one can use the compactness of $S(N_{\mathbb{R}})$ to finish the proof. Otherwise, we increment the dimension of σ_e , using restriction to an appropriate face of \square and induction on dimension. The key idea behind the induction step is to bound the pairs (\square, ψ) appearing in Theorem 2, assuming that \square is a rational polytope. The general case is a combination of the following two examples:

1) If \square is a rational polytope of maximal dimension, property (i) is equivalent to

$$\Delta_{\square}(1) \subseteq \{e \in N_{\mathbb{R}}; \psi(e) \leq 1\}.$$

Here Δ_{\square} is the ample fan in N associated to \square , defined by

$$T_N \text{emb}(\Delta_{\square}) = \text{Proj} \left(\bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap i\square} \mathbb{C}\chi^m \right),$$

and $\Delta_{\square}(1)$ is the set of primitive lattice vectors on the one dimensional cones of Δ . It follows that Δ_{\square} belongs to a finite family of fans, by the compactness of the level sets of ψ .

2) If $\square = \{0\}$, property (i) is equivalent to

$$M \cap \overset{\circ}{\square}_{-\psi} = \{0\}.$$

By (ii) and Kannan-Lovász's effective bound [3] of the width of a convex set in terms of the number of lattice points it contains, there exists $e \in N \setminus 0$ such that $\psi(e) + \psi(-e) \leq C$, where C is a positive constant depending on the dimension of the lattice N only.

Finally, we remark that property (ii) is an “adjoint type” generalization of the weak Fano property of (X, B) in the definition of FGA algebras. It is not clear if it has a non-toric analogue. If $\dim(N) = 2$, property (i) alone implies that \square is a rational polytope. If $\dim(N) \geq 3$, property (ii) is used in the inductive step of the proof.

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