Holger Brenner, University of Sheffield

Grothendieck topologies and ideal closure operations

In this talk I would like to report on a project (in progress, joint work with Manuel Blickle, Essen), intended to relate Grothendieck topologies and closure operations for ideals and submodules. Grothendieck topologies are a generalization of topological spaces forgetting everything except the notion of a covering. There exist in general non-trivial coverings consisting of only one map. This allows still to define sheaves and cohomology, and has led to the construction of the flat and of the etale topology (or site), etale cohomology and the solution to the Weil conjectures in algebraic geometry.

On the other hand, closure operations play an important role in commutative algebra, like the radical, integral closure, tight closure, Frobenius closure, plus closure, solid closure. The basic idea is to understand such closure operations as sheafification in a suitable, non-flat Grothendieck topology, in order to use this highly developed machinery.

In the classical flat Grothendieck topologies, every coherent ideal can be considered as an ideal sheaf in the topology, and the global evaluation gives back the ideal (equivalently, coherent sheaves on affine schemes have trivial cohomology in the flat or in the etale site). In this sense the classical topologies yield the trivial (identical) closure operation. For non-flat Grothendieck topologies however an ideal gives at once only an ideal presheaf, and its sheafification cuts back to an ideal which might be bigger than the original one. So Grothendieck topologies yield closure operations with certain structural properties, e.g. they are persistent.

If, on the other hand, a given closure operation has these structural properties, then it is possible to construct a (minimal) Grothendieck topology yielding back this operation: The affine coverings $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$, which constitutes the topology, are essentially given by the forcing algebras

$$B = A[T_1, \cdots, T_n]/(f_1 T_1 + \cdots + f_n T_n + f)\,,$$

where $f \in (f_1, \cdots, f_n)^c$, c denoting the given closure operation. Forcing algebras appeared first in the work of M. Hochster on solid closure and are fundamental for the geometric understanding of tight closure.

Under the viewpoint of Grothendieck topologies the closure operation itself is believed to be only a tip of an iceberg, while the topology gives at once a new structure sheaf, a cohomology theory (hard to compute), a concept of exactness and of resolution, a concept of stalks. For example, in the tight topology (which is not yet constructed), the stalks are hoped to give big Cohen-Macaulay algebras in a natural way, like the stalks in the etale site give the Henselisation of a local ring, and exactness in the tight topology is related to phantom acyclity.

Special emphasis will be given on the integral closure, where the correspondence is so far best understood. The "natural" Grothendieck topology for the integral closure is given by taking the universally Zariski-submersive mappings as coverings. Zariski-submersive means that a subset with an open preimage must itself be open. This realtionship relies on the fact that both properties can be tested with base change to discrete valuation domains (work of Picavet).