ABSTRACT : PROJECTIONS AND LINEAR SYZYGIES

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Let $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth projective variety embedded by the complete linear system of a very ample line bundle $\mathcal{L}$ on X. One can ask detailed information about defining equations of X, i.e. their syzygies of X. As M. Green defined in [4], we can say that $\mathcal{L}$ satisfies property N_0 if it gives the projectively normal embedding and $\mathcal{L}$ satisfies property N_1 if property N_0 holds and X is cut out by quadrics. In general, $\mathcal{L}$ satisfies property N_p, $p \geq 1$ if X is projectively normal and the projective coordinate ring $S(X)$ of X has the following minimal free resolution of the simplest type as a graded S-module;

$$\to \cdots \to E_p \to E_{p-1} \to \cdots \to E_1 \to S \to S(X) \to 0$$

where $S = \mathrm{Sym}(H^0(\mathcal{L}))$ and $E_i = \bigoplus^{\beta_{i,1}} S(-i-1)$ for all $1 \leq i \leq p$. In other words, property N_p means the minimal free resolution of the homogeneous coordinate ring $S(X)$ of the projectively normal embedding of X is linear until the p-th step.

In this talk, we are interested in geometric and algebraic properties of a very ample subsystems of $H^0(\mathcal{L})$. There are two kinds of subsystems of $H^0(\mathcal{L})$ with respect to projections. One gives an isomorphic outer projection of $X \subset \mathbb{P}(H^0(\mathcal{L}))$ with centers outside $\mathrm{Sec}(X)$ which is not linearly normal and the other gives an inner projection of X which is closely related to the very ample complete linear systems of blow ups of X. In their paper ([5]), S. Kwak and E. Park gave the geometric and syzygetic effects of property N_p of $X \subset \mathbb{P}(H^0(\mathcal{L}))$ to the isomorphic projection of X in $\mathbb{P}(W)$ by a very ample subsystem $W \subset H^0(\mathcal{L})$. More precisely, one can generalize property N_p to property ${N_p}^{S_W}$ for a smooth variety $X \subset \mathbb{P}(W)$ as follows: X satisfies property ${N_p}^{S_W}$ if for $R = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^\ell)$, it has the following minimal free resolution of the simplest type as a graded S_W-module;

$$\to \cdots \to E_p \to E_{p-1} \to \cdots \to E_1 \to S_W \oplus S_W(-1)^t \to R \to 0$$

where $S_W = \mathrm{Sym}(W)$, $t = \mathrm{codim}(W, H^0(\mathcal{L}))$ and $E_i = \bigoplus^{\beta_{i,1}} S_W(-i-1)$ for all $1 \leq i \leq p$. In other words, property ${N_p}^{S_W}$ means the minimal free resolution of R as a graded S_W-module is linear until the p-th step. Recently, Kwak and Park result is generalized to the following ([2]);

Theorem 1. *Let $X \subset \mathbb{P}^r$ be a reduced nondegenerate projective subscheme which satisfies property N_p^S. If $X \subset \mathbb{P}^{r-t}$ is an isomorphic linear projection where $0 \leq t \leq p$, then $X \subset \mathbb{P}^{r-t}$ satisfies property N_{p-t}^S.*

We are also interested in an inner projection of X which is given by subsystem W of $H^0(\mathfrak{L})$ passing through a given point $q \in X$. Let X be a smooth projective variety in $\mathbb{P}(H^0(\mathfrak{L}))$. For a closed point $q \in X$ the inner projection $\pi_q : X \dashrightarrow \mathbb{P}(W)$ defined by $\pi_q(p) = \overline{qp} \cap \mathbb{P}(W), p \neq q$, is a rational map. We can understand this situation in the following diagram; for a blow up $\mathrm{Bl}_q(X)$ of X at q, one has the regular morphism $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to \mathbb{P}(W)$ with a commutative diagram;

$$\begin{array}{ccc} \mathrm{Bl}_q(X) & & \\ \downarrow \sigma & \searrow^{\widetilde{\pi}_q} & \\ X \subset P(H^0(\mathfrak{L})) & \overset{}{\underset{\pi_q}{\dashrightarrow}} & X' = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}(W) \end{array}$$

It is said that X admits an inner projection at a point $q \in X$ if the morphism $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to \overline{\pi_q(X \setminus \{q\})}$ is an embedding, i.e. $\sigma^*\mathfrak{L} - E$ is very ample. When one consider this situation in a projective embedding, we have a nice criterion for $\sigma^*\mathfrak{L} - E$ to be very ample. Note that the inner projection $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to \mathbb{P}(W)$ with center $q \in X$ is a closed embedding if and only if $q \notin \mathrm{Trisec}(X)$ where $\mathrm{Trisec}(X)$ is the union of all trisecant lines ℓ or $\ell \subset X$ ([3]). In this case, $\widetilde{\pi_q}(\mathrm{Bl}_q(X))$ is equal to $\overline{\pi_q(X \setminus \{q\})}$, the Zariski closure of $\pi_q(X \setminus \{q\})$.

With these in mind, we get the following theorem([1]) for an inner projection;

Theorem 2. *Let $X \subset \mathbb{P}(H^0(\mathfrak{L}))$ be a smooth irreducible projective variety. Suppose $\mathfrak{L}$ satisfies property N_p for $p \geq 1$. For any $q \in X \setminus \mathrm{Trisec}(X)$, $\widetilde{\pi_q}(\mathrm{Bl}_q(X)) = \overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W)$ is smooth and satisfies property N_{p-1}, i.e. property N_{p-1} holds for $(\mathrm{Bl}_q(X), \sigma^*\mathfrak{L} - E)$.*

Our method is cohomological, i.e. by using Koszul cohomology technique due to M. Green, we can show $\mathrm{Tor}_i^{S_W}(R', \mathbb{C})_{i+j} = 0, 0 \leq i \leq p$ and $j \geq 2$ for a finitely generated graded S_W module $R' = \bigoplus_{\ell \in \mathbb{Z}} H^0((\sigma^*\mathfrak{L} - E)^\ell)$.

Finally, we talk about some applications of our results to various varieties and their embeddings, for examples the inner projections of Veronese embeddings, Calabi-Yau manifolds, rational surfaces, and adjoint linear series of projective varieties.

References

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