Singularities of maps and birational rigidity

Tommaso de Fernex

"Singularity Theory and Commutative Algebra" – Japan, September 2005

This talk will be devoted to an overview of the methods involved in the study of birational maps between a certain class of algebraic varieties, called Mori fiber spaces. I will explain how the theory of singularities of pairs naturally comes into the picture. All varieties will be defined over the complex numbers.

A *Mori fiber space* is by definition a normal projective variety with terminal, $\mathbb{Q}$-factorial singularities, endowed with an extremal Mori contraction of fiber type.[1] These varieties are one of the two expected "final output" of the Minimal Model Program, the other one consisting of minimal models. As one would like to choose such a variety as the "simplest" object to represent its birational class, it is then natural to inquire about the unicity of such a choice. This led to the notion of birational rigidity.

We say that a Mori fiber space V is *birationally superrigid* if the only birational maps from V to another Mori fiber space are the automorphisms of V. This is a very strong condition, which in particular implies that V is not rational, V is the only Mori fiber space in its birational class, and $\mathrm{Bir}(V) = \mathrm{Aut}(V)$. A slightly weaker notion is that of *birational rigidity*, for which, roughly speaking, the Mori fiber structure of V is required to be "birationally" defined in a unique way.

The first example of birational superrigidity was discovered by Iskovskikh and Manin, with their proof of non-rationality of smooth quartic threefolds in $\mathbb{P}^4$. This also gave a counter-example to a famous problem posed by Lüroth.[2] The method introduced in the proof, known as the *method of maximal singularities*, extends the original ideas of Noether and Fano to effectively bound the singularities of a birational map.

During the talk I will mainly focus on the case of projective Fano hypersurfaces, discussing how the result of Iskovskikh and Manin generalizes to higher dimensions. Considering the problem of birational rigidity for a hypersurface $V \subset \mathbb{P}^N$, it becomes quickly clear that the question is pertinent only when the degree of V is N. The general conjecture, posed by Pukhlikov, is that every smooth hypersurface of degree N in $\mathbb{P}^N$ is birationally superrigid if $N \geq 4$. In a series of publications, starting with Iskovskikh and Manin's contribution, then Pukhlikov's, Cheltsov's, and finally, a paper that I have coauthored with Ein and Mustaţă, this conjecture has been progressively established in a few low dimensional cases. Pukhlikov also found a proof in all dimensions under certain conditions on the equation defining the hypersurface. All together, this is what we know so far:

Theorem. *Let V be a smooth hypersurface of degree N in $\mathbb{P}^N$. If $4 \leq N \leq 12$, then V is birationally superrigid. Moreover, if V is chosen to be sufficiently general in its moduli space, then superrigidity holds for every $N \geq 4$.*

It is interesting to observe that the non-rationality for general V was already known by a result of Kollár, but even this weaker property is unknown to hold for every V.

[1] Fano manifolds with Picard number 1 are, in a trivial way, examples of Mori fiber spaces.

[2] Another counter-example to the Lüroth problem was independently found by Clemens and Griffiths, and more were found shortly afterwards by Artin and Mumford.

Let me now give here a very brief outline of the general method.[3] The basic idea is to assume the existence of a birational map

$$\phi : V \dashrightarrow V', \quad \phi \neq \text{isomorphism},$$

between V and some other Mori fiber space V'. Then one would like to quantify somehow the singularities of the indeterminacy locus of the map, hoping to get a contradiction by considering the degrees of the variety and the equations defining the map. The following is the first key fact.

Noether-Fano Ineqality. *If $B \subset V$ is the subscheme cut by the equations defining V, and r is the degree of such equations, then the pair $(V, \frac{1}{r}B)$ is not canonical.*

The difficulty is to translate the information contained in this statement in terms of more classical notions of singularities that can be more easily compared to the various degrees. Following an idea of Corti, one can tackle this problem in two steps, by reducing first to work with log-canonical thresholds.

Corollary of Shokurov's Connectedness Principle. *If a point p is a center of non canonicity for a pair $(V, \lambda B)$ and $X \subset V$ is a general hyperplane section through p, then the same point is a center of non log-canonicity for the pair $(X, \lambda B|_X)$.*

The second step is to compare log-canonical thresholds with multiplicities. The following was proven by Corti in dimension 2 and then generalized in all dimensions by Ein, Mustaţă and myself.

Theorem. *Let $R = \mathcal{O}_{X,p}$ be a n-dimensional regular local ring, and assume that $\mathfrak{a} \subset R$ is a zero dimensional ideal. Then, denoting by $e(\mathfrak{a})$ the Samuel multiplicity of $\mathfrak{a}$ and by $\operatorname{lc}(\mathfrak{a})$ the log-canonical threshold of $\mathfrak{a}$, we have*

$$e(\mathfrak{a}) \geq (n/\operatorname{lc}(\mathfrak{a}))^n.$$

Finally the circle is complete: we started with a geometric information on V (the existence of a certain rational map) and ended up with a new geometric information on V (the existence of some subscheme with very high multiplicity at some point).

However, although the above inequality suffices to draw the conclusion in the case of quartic threefolds, the gap with the degree of V becomes too large as the dimension increases. A *projection method*, initiated by Pukhlikov and then completed in a joint work by Ein, Mustaţă and myself, leads to the proof up to the case $N = 12$. The main ingredient is the following inequality on log-canonical thresholds.

Theorem. *Let $f : X \to Y$ be a smooth proper morphism of smooth algebraic varieties, and let $Z \subset X$ be a locally complete intersection subscheme, of pure codimension k, such that $\dim Z = \dim Y - 1$ and $f|_Z$ is finite. Then*

$$\operatorname{lc}(Y, f_*[Z]) \leq (\operatorname{lc}(X, Z)/k)^k.$$

At the moment, as N gets larger than 12, one runs into some technical difficulty in controlling the singularities after projection.

[3] The various notions of singularities that appear in this abstract will be introduced during the talk.