CONTACT LOCI AND VALUATIONS

LAWRENCE EIN

This is joint work with R. Lazarsfeld and M. Mustaţă. Let X be a smooth complex variety. Given $m \geq 0$, we denote by

$$X_m \;=\; \mathrm{Hom}\big(\,\mathrm{Spec}\,\mathbb{C}[t]/(t^{m+1})\,,\, X\,\big)$$

the space of m^{th} order arcs on X. Similarly we define the space of formal arcs on X as

$$X_\infty = \mathrm{Hom}\big(\,\mathrm{Spec}\,\mathbb{C}[[t]]\,, X\,\big)$$

Consider now a non-zero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ defining a subscheme $Y \subseteq X$. Given a finite or infinite arc γ on X, the order of vanishing of $\mathfrak{a}$ — or the order of contact of the corresponding scheme Y — along γ is defined in the natural way.[1] For a fixed integer $p \geq 0$, we define the *contact loci*

$$\mathrm{Cont}^p(Y) \;=\; \mathrm{Cont}^p(\mathfrak{a}) \;=\; \big\{\, \gamma \in X_\infty \,|\, \mathrm{ord}_\gamma(\mathfrak{a}) = p\,\big\}.$$

These are locally closed cylinders, i.e. they arise as the common pull-back of the locally closed sets

$$\mathrm{Cont}^p(Y)_m \;=\; \mathrm{Cont}^p(\mathfrak{a})_m \;=_{\mathrm{def}}\; \big\{\, \gamma \in X_m \,|\;\; \mathrm{ord}_\gamma(\mathfrak{a}) = p\,\big\} \tag{1}$$

defined for any $m \geq p$. Let W be the closure of an irreducible component of $\mathrm{Cont}^p(\mathfrak{a})$. We can naturally associate a valuation of the function field of X to W in the following manner. Let f be a nonzero rational function of X. We define

$$\mathrm{val}_W(f) = \mathrm{ord}_\gamma(f) \;\text{ for a general } \gamma \in W.$$

Such a valuation is called a contact valuation. Suppose $\mu : X' \longrightarrow X$ be a proper birational morphism. Assume that E is an irreducible divisor in X'. We can define the valuation associated to E by $\mathrm{val}_E(f) =$ the vanishing order of f along E. A valuation on the function field of X is called a divisoriral valuation if it is of the form $m \cdot \mathrm{val}_E$ for some positive integer m. A basic invariant of val_E from higher dimensional birational geometry is the discrepancy along E which is defined as

$$k_E = \mathrm{val}_E(det(J(\mu)) \;\text{ where } J(\mu) \text{ is the Jacobian matrix of } \mu.$$

k_E is just the coefficient of the relative canonical divisor $K_{X'/X}$ along E.

Theorem A. *Every contact valuation is a divisorial valuation. Conversely, every divisorial valuation can be realized uniquely as a contact valuation.*

[1] Specifically, pulling $\mathfrak{a}$ back via γ yields an ideal (t^e) in $\mathbb{C}[t]/(t^{m+1})$ or $\mathbb{C}[[t]]$, and one sets

$$\mathrm{ord}_\gamma(\mathfrak{a}) \;=\; \mathrm{ord}_\gamma(Y) \;=\; e.$$

(Take $\mathrm{ord}_\gamma(\mathfrak{a}) = m+1$ when $\mathfrak{a}$ pulls back to the zero ideal in $\mathbb{C}[t]/(t^{m+1})$ and $\mathrm{ord}_\gamma(\mathfrak{a}) = \infty$ when it pulls back to the zero ideal in $\mathbb{C}[[t]]$.)

In the above correspondence, suppose that a contact valuation val_W is equal to a divisorial valuation $m \cdot Val_E$. The following theorem relates the geometry between the two valuations.

Theorem B. $\operatorname{codim}(W, X_\infty) = m \cdot (k_E + 1)$.

The above two theorems also hold for singular varieties after some minor modifications using Nash's blow-up and Mather's canonical class.

These results can be used to study singularities of pairs. Let X be a normal $\mathbb{Q}$-Gorenstein complex variety and Y be closed subscheme of X. We also fix a positive number λ. One can study the singularities of the pair $(X, \lambda \cdot Y)$ using log-resolution. Consider a divisorial valuation of the form val_E with center $c_X(E)$ in X. We consider the log discrepancy of $(X, \lambda \cdot Y)$ along E,

$$a(E, X, \lambda \cdot Y) = k_E + 1 - \lambda \cdot \mathrm{val}_E(I_Y),$$

where I_Y is the ideal of Y in X. Suppose that B is a closed subset of X. We can measure the singularities of the pair $(X, \lambda \cdot Y)$ along B using the invariant minimal log-discrepancy.

Definition 0.1. Let $B \subseteq X$ be a nonempty closed subset. The minimal log discrepancy of $(X, \lambda \cdot Y)$ on W is defined by

$$mld(B; X, \lambda \cdot Y) := \inf_{c_X(E) \subseteq W} \{a(E; X, \lambda \cdot Y)\}. \tag{2}$$

Suppose D is a normal effective Cartier divisor in X and B be a closed subset of D. The following theorem is a joint result with Mustaţă. It allows us to compute the minimal log-discrepancy $mld(B; X, D + \lambda \cdot Y)$ using $mld(B; D, \lambda \cdot Y|_D)$.

Theorem C. *Let X be a normal, local complete intersection variety, and Y be a proper closed subscheme of X. Let λ is a positive number. Assume $D \subset X$ is a normal effective Cartier divisor such that $D \not\subseteq Y$, then for every proper closed subset $B \subset D$, we have*

$$mld(B; X, D + \lambda \cdot Y) = mld(B; D, \lambda \cdot Y|_D).$$

The theorem is first proved in the case that X is smooth in a joint paper of Ein, Mustaţă and Yasuda. In general, Kollár, and Shokurov have conjectured that the result is true when X is just normal and $\mathbb{Q}$-Gorenstein (Inversion of Adjunction). See Kollár's article for a discussion of this conjecture and related topics. The following are some geometric applications of the Theorem.

Theorem D. *If X is a normal, local complete intersection variety, and Y is a closed subscheme. Suppose that λ is a positive number then the function $x \longrightarrow mld(x; X, \lambda \cdot Y)$, $x \in X$, is lower semicontinuous.*

Theorem E. *Let X be a normal, local complete intersection variety. X has log canonical (canonical, terminal) singularities if and only if X_m is equidimensional (respectively irreducible, normal) for every m.*

References

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