ASYMPTOTIC INVARIANTS OF LINE BUNDLES

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This is joint work with Lazarsfeld, Mustaţă, Nakamaye and Popa. Let X be a smooth projective variety of dimension n. Denote by $NS(X)_{\mathbf{Q}}$ and $NS(X)_{\mathbf{R}}$ the Nerson-Severi groups of X with rational and real coefficients respectively. In $NS(X)_{\mathbf{R}}$, there are two natural closed cones. These are the nef cone and the cone of pseudo effective divisors. The interior of the nef cone is the ample cone and the interior of the cone of pseudo effective divisors is the cone of big divisors.

It is classical that ample divisors on X satisfy many beautiful geometric, cohomological and numerical properties. By contrast, examples due to Cutkosky and others have led to the impression that the linear series associated with big divisors are in general mired in pathology.

From the recent work of Nakayama, Tsuji, Boucksom-Demailly-Paun-Peternell and ourselves (with Lazarsfeld, Mustaţă Nakamaye and Popa), it has become apparent that arbitrary big divisors in fact display a surprising number of properties analogous to those of ample line bundles.

If D is a divisor on X with $\mathbf{Q}$-coefficients, then the *stable base locus* $\mathbf{B}(D)$ is defined to be the common base locus of the complete linear series $|mD|$ for sufficiently divisible m. Understanding $\mathbf{B}(D)$ is difficult in general, since this locus does not depend only on the numerical class of D. It was discovered by Nakamaye that an upper approximation, the *augmented base locus* of D, behaves much better than $\mathbf{B}(D)$ itself. This is defined as $\mathbf{B}_+(D) := \mathbf{B}(D - A)$, where A is any "small" ample $\mathbf{Q}$-divisor, and is a proper subset of X if and only if D is big. A similar definition can be given for any $\mathbf{R}$-divisor. The variation of $\mathbf{B}_+$ is shown to give an interesting decomposition of the cone of big divisors $\mathrm{Big}(X)$; for a detailed study cf. [ELMNP2]. In the case of *nef* divisors, it was shown in [Na1] that $\mathbf{B}_+(D)$ can be described numerically by the vanishing of intersection numbers:

$$\mathbf{B}_+(D) = \bigcup_{(D^{\dim V} \cdot V)=0} V, \tag{1}$$

Recall that if L is a line bundle on X, then its volume is an asymptotic invariant naturally associated to the Riemann-Roch problem. Specifically, it measures the rate of growth of the sections of multiples of L:

$$\mathrm{vol}_X(L) := \limsup_{m\to\infty} \frac{h^0(X, L^m)}{m^n/n!}.$$

More generally, suppose that V is a subvariety of dimension $d \geq 1$ in X. The *restricted volume* of L along V measures the rate of growth of the sections of multiples of L on V

that can be lifted to X:

$$\mathrm{vol}_{X|V}(L) := \limsup_{m\to\infty} \frac{\dim_{\mathbf{C}} \mathrm{Im}(H^0(X, L^m) \to H^0(V, L^m|_V))}{m^d/d!}.$$

The definition extends in a natural way to $\mathbf{Q}$-divisors and it is easy to see that if $V \not\subseteq \mathbf{B}_+(D)$, then $\mathrm{vol}_{X|V}(D) > 0$.

If L is ample, then $\mathrm{vol}_X(L) = c_1(L)^n$. One gets the following geometric interpretation. If $D_1, \ldots, D_n$ are general divisors in $|mL|$, then

$$\mathrm{vol}_X(D) = \frac{\#(D_1 \cap \ldots \cap D_n)}{m^n}.$$

The following theorem is the analogous result for big divisors.

Theorem A. *Let D be a $\mathbf{Q}$-divisor and V a subvariety of dimension $d \geq 1$ such that $V \not\subseteq \mathbf{B}_+(D)$. If we consider, for m divisible enough, general divisors $D_1, \ldots, D_d$ in $|mD|$, then we have*

$$\mathrm{vol}_{X|V}(D) = \lim_{m\to\infty} \frac{\#(V \cap D_1 \cap \ldots \cap D_d \smallsetminus \mathbf{B}(D))}{m^d}.$$

The following is analogue of Nakamaye's theorem for big divisors.

Theorem B. *If D is a $\mathbf{Q}$-divisor on X, then $\mathbf{B}_+(D)$ is the union of all positive dimensional subvarieties V such that $\mathrm{vol}_{X|V}(D) = 0$.*

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