Toward higher dimensional Auslander-Reiten theory

Osamu Iyama

Let G be a finite subgroup of $\mathrm{SL}_d(k)$ (char $k = 0$), $S := k[[x_1, .., x_d]]$, S^G an invariant subring of G and $S * G$ the skew group ring.

For the case $d = 2$, it is well-known that S^G is representation-finite, i.e. S^G has only finitely many isoclasses of indecomposable maximal Cohen-Macaulay modules. More strongly, Auslander found that the McKay quiver of G coincides with the Auslander-Reiten quiver of S^G.

For the case $d > 2$, S^G is never representation-finite. The aim of this talk is to study the category $\mathrm{CM}S^G$ of maximal Cohen-Macaulay S^G-modules from the viewpoint of 'higher dimensional' Auslander-Reiten theory.

Let us recall a classical theorem of Auslander in the representation theory of finite-dimensional algebras: There exists a bijection between Morita-equivalence classes of representation-finite finite-dimensional algebras and those of finite-dimensional algebras with global dimension at most two and dominant dimension at least two. This theorem can be regarded as a starting point of Auslander-Reiten theory historically.

For a non-negative integer n, we introduce the concept of *maximal n-orthogonal subcategory* of $\mathrm{mod}\Lambda$ (resp. $\mathrm{CM}\Lambda$) for a finite-dimensional algebra (resp. Cohen- Macaulay ring) Λ. It can be shown that there exists a bijection between equivalence classes of maximal $(n-2)$- orthogonal subcategories and Morita-equivalence classes of finite-dimensional algebras with global dimension at most n and dominant dimension at least n.

Thus it would be natural to study 'higher dimensional' Auslander-Reiten theory on maximal n-orthogonal subcategories: We can obtain 'higher' Auslander-Reiten duality theorem and the existence theorem of 'higher' Auslander-Reiten sequences. Moreover, $\mathrm{CM}S^G$ has a 'canonical' maximal $(d-2)$- orthogonal subcategory $\mathrm{add}S$, whose Auslander-Reiten quiver coincides with the McKay quiver of G. But $\mathrm{CM}S^G$ has many other maximal $(d-2)$-orthogonal subcategoires.

In this talk, we will study how to get *all* maximal $(d-2)$-orthogonal subcategories of $\mathrm{CM}S^G$. Thanks to '$(d-1)$- Calabi-Yau' property of $\mathrm{CM}S^G$, we can show that, using 'higher' Auslander-Reiten sequences in a maximal orthogonal subcategory, one can get other maximal orthogonal subcategories [joint work with Y. Yoshino]. This process can be regarded as an analogy of *mutation*, which is well-known in representation theory of algebras due to Rudakov (vector bundles on $\mathbf{P}_n$), Riedtmann-Schofield (classical tilting modules), Buan-Marsh-Reineke-Reiten-Todorov (cluster categories) and Geiss-Leclerc-Shr'oer (preprojective algebras). It is natural to ask whether one can get all maximal $(d-2)$-orthogonal subcategories of $\mathrm{CM}S^G$ by successive mutations. In the talk of Yoshino, this is shown to be true for the case $G = \langle \mathrm{diag}(\omega, \omega, \omega) \rangle$ $(\omega^3 = 1)$.

We also discuss the relationship among maximal $(d-2)$- orthogonal subcategories of $\mathrm{CM}S^G$ and the following objects [joint work with I. Reiten].

(i) Van den Bergh's non-commutative crepant resolutions of S^G,

(ii) tilting $S * G$-modules,

(iii) Fomin-Zelevinsky mutation.

Especially, we will show that certain non-commutative analogy of Bondal-Orlov conjecture due to Van den Bergh on derived equivalences of crepant resolutions is true for three dimensional isolated singularities.