On the canonical ring of a surface

Kazuhiro Konno (Osaka University)

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Let S be a non-singular, projective, minimal surface of general type defined over the complex number field $\mathbb{C}$. The canonical ring of S is of course the graded $\mathbb{C}$-algebra

$$R(S, K_S) = \bigoplus_{m=0}^{+\infty} H^0(S, mK_S).$$

This naive object has been studied by several authors, after the success of Kodaira and Bombieri on the pluri-canonical systems. Let me recall some important results obtained so far: $R(S, K_S)$ is

- generated in degrees ≤ 5, when $p_g(S) > 0$ (Ciliberto [1], ’83),
- generated in degrees ≤ 3 and related in degrees ≤ 6, when $p_g(S) \geq 2$, $q(S) = 0$, $K_S^2 \geq 3$ and the canonical model has an irreducible canonical curve (Reid [7], ’88),
- generated in degrees ≤ 4, when $p_g(S) = 0$ and $2K_S$ is free (Mendes Lopes [6], ’97).

The difficulty arises from the simple fact that $|K_S|$ may well have base points or fixed components, unlike curves. Hence we must accept it as a common rule that $R(S, K_S)$ needs a generator in degree 3. Then it is too much to hope that the presence of a primitive generator in “high” degree forces S to have a “special” linear system. This may be enough to disappoint everyone. However, it is not a tiny thing to know how the algebraic structure of $R(S, K_S)$ reflects on the geometry of S, and vice versa.

I talk about my recent results on the degree bounds for generators and relations. A remarkable theorem essentially due to Francia [3] and Reider [8] states that $2K_S$ is free except possibly when $p_g(S) = 0$ and $K_S^2 \leq 4$. My first result is a simple application of the theorem via the Koszul cohomology [2], and is nothing more than what one naturally expects after Ciliberto’s bound for generators.

Theorem 1 *Let S be a minimal algebraic surface of general type such that $2K_S$ is free. Then $R(S, K_S)$ is generated in degrees ≤ 5 and related in degrees ≤ 10. If furthermore $q(S) = 0$, then $R(S, K_S)$ is generated in degrees ≤ 4 and related in degrees ≤ 8 except when $(p_g(S), K_S^2) = (2, 1)$.*

A numerically 2-connected curve E is called a *Francia cycle* if either (i) $p_a(E) = 1$, $E^2 = -1$, or (ii) $p_a(E) = 2$, $E^2 = 0$. It is known that $R(S, K_S)$ needs a generator in degree 4 (and a relation in degree 8) if the fixed part of $|K_S|$ contains a Francia cycle. Many such surfaces are known to exist. Hence Theorem 1 cannot be improved further (at least for regular surfaces) in this sense. Nevertheless, it is an interesting problem to detect whether any criminal is a Francia cycle in the fixed part, as in the case of fibrations [4] (see also Theorem 5 below). Here is my partial answer:

Theorem 2 *Let S be a minimal algebraic surface of general type with $p_g(S) \geq 2$, $q(S) = 0$ and $K_S^2 \geq 3$. Let $|K_S| = |M| + Z$ be the decomposition into its variable and fixed parts, and suppose that $|M|$ has an irreducible member. If one of the following conditions* (1), (2) *is satisfied, then $R(S, K_S)$ is generated in degrees ≤ 3 and related in ≤ 6.*

(1) $H^0(Z, K_Z) = 0$ (*possibly* $Z = 0$).

(2) *Z does not contain any Francia cycles and decomposes as $Z = \Delta + \Gamma_1 + \cdots + \Gamma_n$, where*

(a) *Δ is an effective divisor with $K_S\Delta = 0$ (possibly $\Delta = 0$), $\mathrm{Supp}(\Delta) \cap \mathrm{Supp}(Z - \Delta) = \emptyset$, and*

(b) *for each $i \in \{1, \ldots, n\}$, Γ_i is a chain connected curve such that $K_S\Gamma_i > 0$, $\mathcal{O}_{\Gamma_i}(-\Gamma_i)$ is nef, $\Gamma^2 \leq 0$ holds for any subcurve $\Gamma \leq \Gamma_i$ and, when $j \neq i$, $\mathcal{O}_{\Gamma_i}(\Gamma_j)$ is numerically trivial.*

My proof depends on the hyperplane-section principle which does not work well without assuming $q(S) = 0$. $|M|$ contains an irreducible member unless the canonical map is composed of a pencil; so it may be harmless to assume it. If Z supports exceptional sets of rational singular points at most, then $h^0(Z, K_Z) = 0$; so (1) is an extension of Reid's theorem. An effective divisor D is *chain connected* if either it is irreducible, or for any proper subcurve Γ there exists an irreducible component C of Γ such that $(D - \Gamma)C > 0$; hence the fundamental cycle of a normal surface singularity is automatically chain connected and serves an example of Γ_i in (b). Though I do hope that the technical assumptions in (2) will be removed eventually, it would be worth stating here the following theorem which is crucial in the proof of Theorem 2, and which is a generalization of a theorem in [5] to the negative semi-definite case.

Theorem 3 *Let D be a chain connected curve on a non-singular surface S with $K_S^2 > 0$ such that K_S and $-D$ are both nef on D. Assume furthermore that $\Delta^2 \leq 0$ holds for any subcurve Δ of D. Let L_α ($\alpha = 1, 2$) be a line bundle on D such that $L_\alpha - 2K_S$ is nef. Then the multiplication map*

$$H^0(D, L_1) \otimes H^0(D, L_2) \to H^0(D, L_1 + L_2)$$

is surjective, unless there exists a subcurve $E \leq D$ such that $L_1 \equiv L_2 \equiv 2K_S$ on E and either $p_a(E) = 1$, $E^2 = -1$, or $p_a(E) = 2$, $E^2 = 0$. Here, the symbol $\equiv$ means the numerical equivalence.

The method also applies to the local and the semi-global cases for bounding the degrees of relations. The following two results answer Reid's 1–2–3 conjecture for the corresponding algebras. As is well-known, the bound for generators is due to Laufer [5] in the case of singularities.

Theorem 4 *The relative canonical algebra for a normal surface singularity is generated in degrees* ≤ 3 *and related in degrees* ≤ 6*.*

Theorem 5 *The relative canonical algebra for a relatively minimal fibration of curves of genus* ≥ 2 *is generated in degrees* ≤ 4 *and related in degrees* ≤ 8*. If furthermore there are no multiple fibres whose canonical system has a* (-1) *elliptic cycle as a fixed component, then the algebra is generated in degrees* ≤ 3 *and related in degrees* ≤ 6*.*

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