Resolution of singularities and the positivity conjecture of Serre

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Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of a regular local ring $(A, \mathfrak{m})$ satisfying (1) $\mathfrak{p} + \mathfrak{q}$ is an $\mathfrak{m}$-primary ideal, (2) $\dim A/\mathfrak{p} + \dim A/\mathfrak{q} = \dim A$. In [3], Serre conjectured that the intersection multiplicity between $\mathrm{Spec}(A/\mathfrak{p})$ and $\mathrm{Spec}(A/\mathfrak{q})$ is positive, i.e.,

$$\chi_A(A/\mathfrak{p}, A/\mathfrak{q}) := \sum_{i \geq 0} (-1)^i \ell_A(\mathrm{Tor}_i^A(A/\mathfrak{p}, A/\mathfrak{q})) > 0.$$

Let $\mathfrak{p}^*$ be the kernel of the natural ring homomorphism $G_A(\mathfrak{m}) \to G_{A/\mathfrak{p}}(\mathfrak{m}{\cdot}A/\mathfrak{p})$. Sometimes $\mathfrak{p}^*$ is called the *initial ideal* of $\mathfrak{p}$. Recently, Dutta [1] proved the inequality $\chi_A(A/\mathfrak{p}, A/\mathfrak{q}) \geq e_{\mathfrak{m}}(A/\mathfrak{p}) \times e_{\mathfrak{m}}(A/\mathfrak{q})$ if $\mathrm{Proj}(G_A(m)/\mathfrak{p}^* + \mathfrak{q}^*)$ is a finite set. Here, $e_{\mathfrak{m}}(A/\mathfrak{p})$ denotes the multiplicity of $A/\mathfrak{p}$ at $\mathfrak{m}$. There are two key points in his proof. The first one is to use the formula

$$\chi_A(A/\mathfrak{p}, A/\mathfrak{q}) = e_{\mathfrak{m}}(A/\mathfrak{p}) \times e_{\mathfrak{m}}(A/\mathfrak{q}) + \eta_*(\sum_i (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_H}(\mathcal{O}_V, \mathcal{O}_W)])$$

as in Example 20.4.3 in [2]. Here, $\pi : H \to \mathrm{Spec}(A)$ is the blow-up at $\mathfrak{m}$, and V (resp. W) is the strict transform of $\mathrm{Spec}(A/\mathfrak{p})$ (resp. $\mathrm{Spec}(A/\mathfrak{q})$). Set $\eta = \pi|_{\pi^{-1}(\mathfrak{m})} : \pi^{-1}(\mathfrak{m}) \to \mathrm{Spec}(A/\mathfrak{m})$. We denote η_* the induced push-forward map $G_0(\pi^{-1}(\mathfrak{m})) \to G_0(A/\mathfrak{m})$. The second key point is to use Gabber's non-negativity theorem.

The formula as above (Example 20.4.3 in [2]) is proved by blowing up the affine scheme $\mathrm{Spec}(A[T])$ at $(\mathfrak{m}, T)$.

The aim of my talk is to show that, if there exists *a birational map good enough* (other than the blowing-up of $\mathrm{Spec}(A[T])$ at $(\mathfrak{m}, T)$), then Serre's positivity conjecture is true. It is proved in the same way as Dutta's method.

The following is a problem on the existence of *a birational map good enough* that implies the positivity conjecture.

Problem 1 Let P and Q be prime ideals of a regular local ring $(B, \mathfrak{n})$ satisfying (1) $\mathrm{ht}(P) + \mathrm{ht}(Q) = \dim(B) - 1$, (2) $B/\sqrt{P+Q}$ is a discrete valuation ring. Do there exist an ideal I of B that satisfies the following four conditions?

1. $\tilde{X} := \mathrm{Proj}(B[It])$ is regular.
2. $I \not\subset \sqrt{P+Q}$.
3. Let $\tilde{Y} = \mathrm{Proj}(B/P[(I{\cdot}B/P)t])$, $\tilde{Z} = \mathrm{Proj}(B/Q[(I{\cdot}B/Q)t])$, and $\varphi : \tilde{X} \to \mathrm{Spec}(B)$ be the blow-up along I. Then, $\tilde{Y} \cap \tilde{Z} \cap \varphi^{-1}(\mathfrak{n})$ is a finite set.

4. There exists an irreducible component E of $\varphi^{-1}(\mathfrak{n})$ such that $\mathrm{codim}_{\tilde{X}} E = 1$ and $\tilde{Y} \cap \tilde{Z} \cap \mathrm{reg}(E) \neq \emptyset$.

I do not know if Problem 1 is true or not even in the case where $B = \mathbb{C}[s,t,u]_{(s,t,u)}$ and P and Q are principal ideals.

Putting $B = A[T]_{(\mathfrak{m},T)}$, $P = \mathfrak{p}B$ and $Q = \mathfrak{p}B$, we have the following theorem.

Theorem 2 If Problem 1 is true, then Serre's positivity conjecture is true.

Problem 1 seemes to be very difficult since existence of resolutions of singularity is still a open problem.

Next we weaken the assumption of birationality, i.e., we replace birationality with generically finiteness.

Problem 3 Let P and Q be prime ideals of a regular local ring $(B, \mathfrak{n})$ satisfying (1) $\mathrm{ht}(P) + \mathrm{ht}(Q) = \dim(B) - 1$, (2) $B/\sqrt{P+Q}$ is a discrete valuation ring. Do there exist proper surjective and generically finite map $\varphi : W \to \mathrm{Spec}(B)$ that satisfies the following four conditions?

1. W is regular.
2. There exists an openset U of $\mathrm{Spec}(B)$ such that $U \ni \sqrt{P+Q}$ and $\varphi^{-1}(U) \to U$ is finite.
3. There exist closed subschemes $\tilde{Y}$ and $\tilde{Z}$ such that (1) $\varphi(\tilde{Y}) = V(P)$, (2) $\varphi(\tilde{Z}) = V(Q)$, (3) $\varphi(\tilde{Y} \cap \tilde{Z}) = V(P+Q)$, and (4) $\tilde{Y} \cap \tilde{Z} \cap \varphi^{-1}(\mathfrak{n})$ is a finite set.
4. There exists an irreducible component E of $\varphi^{-1}(\mathfrak{n})$ such that $\mathrm{codim}_W E = 1$ and $\tilde{Y} \cap \tilde{Z} \cap \mathrm{reg}(E) \neq \emptyset$.

Theorem 4 If Problem 3 is true, then Serre's positivity conjecture is true.

We give some examples in my talk.

References

[1] S. P. Dutta, *A special case of positivity II*, preprint.

[2] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, New York, 1984.

[3] J-P. Serre, *Algèbre locale. Multiplicités*, Lect. Note in Math., vol. 11, Springer-Verlag, Berlin, New York, 1965.