Counterexamples to the Fourteenth Problem of Hilbert

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Let K be a field, $K[X] = K[X_1, \dots, X_n]$ the polynomial ring in n variables over K for some $n \in \mathbf{N}$, and $K(X)$ the field of fractions of $K[X]$. Assume that L is a subfield of $K(X)$ containing K. Then, the Fourteenth Problem of Hilbert asks whether the K-subalgebra $L \cap K[X]$ of $K[X]$ is finitely generated. Zariski [17] showed in 1954 that the answer to this problem is affirmative if the transcendence degree $\operatorname{trans.deg}_K L$ of L over K is at most two, while Nagata [15] gave the first counterexample in 1958 in the case where $n = 32$ and $\operatorname{trans.deg}_K L = 4$. In 1990, Roberts [16] constructed a counterexample of different type when $n = 7$ and $\operatorname{trans.deg}_K L = 6$. Following Nagata and Roberts, several new counterexamples have been constructed. We also developed a powerful theory on the Fourteenth Problem of Hilbert by generalizing the construction of Roberts. In this talk, we give various kinds of new counterexamples obtained by applying our theory.

After giving the first counterexample, Nagata presented a question whether there exists a counterexample to the Fourteenth Problem of Hilbert when $\operatorname{trans.deg}_K L = 3$, because the answer had been known to be affirmative if $\operatorname{trans.deg}_K L \le 2$ due to Zariski. This was a longstanding open question, but we finally answer it by giving counterexamples (cf. [9]). Concerning the dimension n of $K[X]$, counterexamples have been found for $n \ge 5$, while there exists no counterexamples when $n \le 2$ due to Zariski. The cases where $n = 3, 4$ were open, but we also give counterexamples in these cases (cf. [9], [10]). Thereby, the Fourteenth Problem of Hilbert is settled for all $\operatorname{trans.deg}_K L$ and n at last.

The problem of finite generation of the kernel of a derivation is an important special case of the Fourteenth Problem of Hilbert. Let D be a derivation of $K[X]$ over K, i.e., a K-linear map $K[X] \to K[X]$ satisfying $D(fg) = D(f)g + fD(g)$ for $f, g \in K[X]$. Then, D extends uniquely to a derivation of $K(X)$. The kernels $K[X]^D$ and $K(X)^D$ of D and its extension to $K(X)$ are a K-subalgebra of $K[X]$ and a subfield of $K(X)$ containing K, respectively. Since $K[X]^D = K(X)^D \cap K[X]$, the problem of finite generation of $K[X]^D$ is a kind of the Fourteenth Problem of Hilbert.

The result of Zariski implies that $K[X]^D$ is always finitely generated if $n \le 3$. On the other hand, Derksen [3] first showed the existence of D for which $K[X]^D$ is not finitely generated when $n = 32$ by using the counterexample of Nagata. In 1994, Deveney-Finston [4] described the counterexample of Roberts by using a derivation of $K[X]$ for

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$n = 7$. For $n = 6$ and $n = 5$, Freudenburg [6] and Daigle-Freudenburg [1] gave D for which $K[X]^D$ is not finitely generated, respectively. Thus, the problem of finite generation of $K[X]^D$ was open only for $n = 4$. We give a negative answer to this open case by showing that our counterexamples for $n = 4$ can be realized as the kernel of a derivation (cf. [11]).

To explain this, we give an outline of the construction of our counterexamples above. First, we recall some notions on derivations. Let D be a derivation of $K[X]$. Then, D is said to be *triangular* if $D(X_i)$ is in $K[X_1, \ldots, X_{i-1}]$ for each i. We say that D is *locally nilpotent* if, for each $f \in K[X]$, there exists l such that $D^l(f) = 0$. Note that a triangular derivation is locally nilpotent. An element $g \in K[X]$ is said to be a *slice* of D if $D(g) = 1$. Assume that D is locally nilpotent and having a slice g. We define

$$\sigma_g^D(f) = \sum_{i=0}^{l-1} \frac{D^i(f)}{i!}(-g)^i$$

for each $f \in K[X]$, where $l \in \mathbf{N}$ such that $D^l(f) = 0$. Then, $\sigma_g^D(f)$ is in $K[X]^D$, since

$$D(\sigma_g^D(f)) = \sum_{i=0}^{l-1} \left(\frac{D^{i+1}(f)}{i!}(-g)^i - i\frac{D^i(f)}{i!}(-g)^{i-1} \right) = \frac{D^l(f)}{(l-1)!}(-g)^{l-1} = 0.$$

It is well-known that $K[X]^D = K[\sigma_g^D(X_1), \ldots, \sigma_g^D(X_n)]$. Now, let $K[Y] = K[Y_1, \ldots, Y_m]$ be the polynomial ring in m variables for some $m \in \mathbf{N}$, and $\tau : \mathbf{Z}^m \to \mathbf{Z}^n$ a $\mathbf{Z}$-linear map. We define a homomorphism $h_\tau : K[Y] \to K(X)$ of K-algebras by $h_\tau(Y^b) = X^{\tau(b)}$ for each b. Here, we denote $X^a = X_1^{a_1} \cdots X_n^{a_n}$ and $Y^b = X_1^{b_1} \cdots Y_n^{b_m}$ for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$. For a locally nilpotent derivation D of $K[Y]$, we define $L(D, \tau)$ to be the field of fractions of $h_\tau(K[Y]^D)$. Then, we may obtain a great variety of counterexamples to the Fourteenth Problem of Hilbert as $L(D, \tau)$ for suitable D and τ. Our counterexample for $n = 4$ can be obtained as $L(D, \tau)$ for a triangular derivation D of $K[Y]$ for $m = 4$ with $D(Y_i) \neq 0$ for each i and an suitable injection τ.

Although $L(D, \tau)$ is defined by using the kernel of a derivation, $L(D, \tau) \cap K[X]$ is not necessarily equal to the kernel of a derivation of $K[X]$. We give the following sufficient condition on D and τ for $L(D, \tau) \cap K[X]$ to be the kernel of a derivation of $K[X]$ (cf. [11]). Assume that τ is injective. Then, $L(D, \tau) \cap K[X]$ is equal to the kernel of some derivation of $K[X]$ if and only if $\mathbf{Z}^n \cap \sum_{i \in I_D} \mathbf{R}\tau(\mathbf{e}_i) = \sum_{i \in I_D} \mathbf{Z}\tau(\mathbf{e}_i)$, where $I_D = \{i \mid D(Y_i) = 0\}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the coordinate unit vectors of $\mathbf{R}^n$. By this criterion, we know that our counterexample for $n = 4$ can be realized as the kernel of a derivation.

When D is locally nilpotent, the problem of finite generation of $K[X]^D$ for $n = 4$ has not been settled yet. The derivations given by Deveney-Finston, Freudenburg, and Daigle-Freudenburg are in fact triangular derivations. Hence, the problem has negative answers for $n \geq 5$. However, our derivation D of $K[X]$ for $n = 4$ is not locally nilpotent. Furthermore, $K[X]^D$ is not contained in $K[X]^{D'}$ for any locally nilpotent derivation D' of $K[X]$. In 2001, Daigle-Freudenburg [2] showed that the kernel of a triangular derivation of $K[X]$ is always finitely generated when $n = 4$. So, we must consider locally nilpotent derivations whose kernels are not equal to the kernels of any triangular derivations.

Towards the solution of this open problem, the study of locally nilpotent derivations of $K[X]$ of rank n is of great important. Here, the *rank* of D is by definition the least integer r for which there exist $f_1, \ldots, f_n \in K[X]$ with $K[f_1, \ldots, f_n] = K[X]$ such that $D(f_i) = 0$ for $i = 1, \ldots, n-r$. The rank of a triangular derivation of $K[X]$ is known to be less than n if $n \geq 2$. Hence, the kernel of a locally nilpotent derivation D of $K[X]$ of rank n is never isomorphic to the kernel of any triangular derivation of $K[X]$ via an automorphism of $K[X]$ when $n \geq 2$. Although Freudenburg [5] has already given an example of a locally nilpotent derivation of $K[X]$ of rank n, the construction of such derivations is quite difficult in general. So, finite generation of their kernels has not been studied. By applying our theory, we construct the first example of a locally nilpotent derivation of $K[X]$ of rank n whose kernel is not finitely generated for each $n \geq 5$ (cf. [14]).

As mentioned, a locally nilpotent derivation of $K[X]$ with a slice is generated by at most n elements. So, van den Essen posed a question whether the kernel of a derivation of $K[X]$ with a slice is always finitely generated, and also conjectured that the answer is no. We show that the conjecture is true by giving an explicit example.

We also have a condition for finite generation of the kernel of a derivation of $K[X]$. For a Laurent polynomial f in $X_1, \ldots, X_n$, we define $\mathrm{supp}(f)$ to be the set of $a \in \mathbf{Z}^n$ such that the monomial X^a appears in f with nonzero coefficient. Let D be a derivation D of $K[X]$. Then, define the *support* $\mathrm{supp}(D)$ of D by

$$\mathrm{supp}(D) = \bigcup_{i=1}^{n} \mathrm{supp}(X_i^{-1} D(X_i)).$$

We call the convex hull $\mathrm{New}(D)$ of $\mathrm{supp}(D)$ in $\mathbf{R}^n$ the *Newton polytope* of D. The dimension of $\mathrm{New}(D)$ is defined as the dimension of the $\mathbf{R}$-vector subspace of $\mathbf{R}^n$ generated by $a - b$ for $a, b \in \mathrm{New}(D)$. Then, our result is that $K[X]^D$ is finitely generated if the dimension of $\mathrm{New}(D)$ is at most two (cf. [7]). We note that there exists D for which $K[X]^D$ is not finitely generated and the dimension of $\mathrm{New}(D)$ is three.

Finally, we discuss the case where $K(X)$ is algebraic over L. In this case, there are only few studies so far. It had even been unknown whether there existed a counterexample until when we gave one for $n = 3$ (cf. [10]). Here, due to Zariski, $K(X)$ is necessarily algebraic over L if $n = 3$ and $L \cap K[X]$ is not finitely generated. For our counterexamples in [10], the extension degrees $[K(X) : L]$ of $K(X)$ over L can only be certain even numbers not less than 22. However, by using a different construction, we give a counterexample with $[K(X) : L] = d$ for each $d \geq 3$ when $n = 3$ (cf. [12]). We remark that the case where $[K(X) : L] = 2$ is open for any $n \geq 3$.

Assume that a finite group G acts on $K(X)$. Then, $K(X)$ is algebraic over the subfield $K(X)^G$ of G-invariant elements of $K(X)$. We define $K[X]^G = K(X)^G \cap K[X]$. By Noether's theorem, $K[X]^G$ is finitely generated if $\sigma \cdot f$ is in $K[X]$ for each $\sigma \in G$ and $f \in K[X]$. However, finite generation of $K[X]^G$ was unknown in general. Using our theory, we give examples of actions of G on $K(X)$ for which $K[X]^G$ are not finitely generated (cf. [13]). We may construct such actions for each finite group containing at least three elements.

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