VERY AMPLE LINE BUNDLES ON REGULAR SURFACES OBTAINED BY PROJECTION

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We work over an algebraically closed field $k$ of arbitrary characteristic. Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface with very ample line bundle $L := \mathcal{O}_X(1)$. For points $x_1, \ldots, x_m \in X$ of $X$, consider the blowing up $\sigma: \hat{X} \rightarrow X$ of $X$ at $x_1, \ldots, x_m$ with the exceptional divisors $E_1, \ldots, E_m$ and the line bundle $\hat{L} = \sigma^* L \otimes \mathcal{O}_X(-E_1 - \cdots - E_m)$ on $\hat{X}$. The problem of finding the condition for $\hat{L}$ to be very ample in terms of the configuration of points $x_1, \ldots, x_m \in X$ is considered by many authors ([Be],[CE],[C],[DH],[L]). This problem arises, for example, in the classification of surfaces by sectional genus (see, for example [I], [L]). In the classification of Del Pezzo surfaces, it is well-known that for $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(3)$, $\hat{L}$ is very ample if and only if $x_1, \ldots, x_m$ are distinct points with $m \leq 6$ such that no three points of $x_1, \ldots, x_m$ are collinear and no six points of them lie on conic in $\mathbb{P}^2$. Here along this line, we give this type of condition for surfaces with $h^1(\mathcal{O}_X) = 0$.

**Theorem 1.** Let $X \subseteq \mathbb{P}^N$ ($N = d - g + 1$) be a nondegenerate, linearly normal, smooth, projective variety of dimension 2, degree $d$, sectional genus $g$, and irregularity $h^1(\mathcal{O}_X) = 0$. Let $x_1, \ldots, x_m$ be distinct $m$ points of $X$ for $m \leq d - 2g - 1$. Let $\sigma: \hat{X} \rightarrow X$ be a blowing up at $x_1, \ldots, x_m$. Then $\hat{L} = \sigma^* \mathcal{O}_X(1) \otimes \mathcal{O}_X(-E_1 - \cdots - E_m)$ is very ample if and only if for all $l$ with $1 \leq l \leq m$, any $l$ points of $\{x_1, \ldots, x_m\}$ do not lie on any rational normal curve of degree $l$ on $X$.

The key ingredient of the proof is the *inner projection*, that is the linear projection from a point of $X$, which appears in the definition of blowing up of a point of $X$. By using this, for $m = 1$, $\hat{L}$ is very ample if and only if there is no line through $x_1$ meeting $X$ in 3 points counted with multiplicity. We apply this argument successively, based on the following lemma.

**Lemma 2.** Let $X$ be as in Theorem.

1. Assume that $d = \deg X \geq 2g + 2$. If $\ell \subseteq \mathbb{P}^N$ is a line not lying on $X$, then $l(X \cap \ell) \leq 2$.

2. If $D \subseteq X$ is an irreducible reduced curve of degree $m$ with $m \leq d - 2g$, then $D \cdot E \leq 1$ for a line $E$ on $X$ with $D \neq E$.

As an application of Theorem, we will give a necessary and sufficient condition for line bundles to be very ample which appear in the classification by the sectional genus due to Ionescu [I, (3.1) and (4.1)]. We deal with the following three cases:

(A) $(X, L)$ is isomorphic to $F_e, \ H_e = 2C_0 + (3 + e)F$, $e = 0, 1, 2$, or to the blowing-up $\sigma$ of one of these with center $k \leq 7$ points lying on different fibres, $H = \sigma^*(H_e) - E_1 - \cdots - E_k$; moreover $5 \leq d \leq 12$. $\Rightarrow g = 2: d - 2g - 1 = 12 - 5 = 7$.

(B) $(X, L)$ is isomorphic to $F_e, \ H_e = 2C_0 + (4 + e)F$, $e = 0, 1, 2, 3$, or to the blowing-up $\sigma$ of one of these with center $k \leq 9$ points lying on different fibres, $H = \sigma^*(H_e) - E_1 - \cdots - E_k$; moreover $7 \leq d \leq 16$. $\Rightarrow g = 3: d - 2g - 1 = 16 - 7 = 9$.

(C) $\mathbb{P}^2, \ H = 4L$ where $L$ is a line, or a blowing-up $\sigma$ of it with center $k \leq 10$ ordinary points, $H = \sigma^*(4L) - E_1 - \cdots - E_k$; moreover $6 \leq d \leq 16$. $\Rightarrow g = 3: d - 2g - 1 = 16 - 7 = 9$.

For example, in the third case (C), we have the following:

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Theorem 3. Let $\sigma : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at $x_1, \ldots, x_k$ with $k \leq 9$ with the exceptional divisors $E_1, \ldots, E_k$. Let $\mathcal{L}$ be the line bundle $\sigma^* \mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\hat{\mathbb{P}}^2}(-E_1 - \cdots - E_k)$. Then the following are equivalent:

1. $\mathcal{L}$ is very ample.
2. For every integer $l$ and divisor $B$ in (3.1), no distinct $l$ points of $\{x_1, \ldots, x_k\}$ lie on any curve linearly equivalent to $B$.
3. For every integer $l$ and divisor $B$ in (3.1), and for every distinct $l$ points $\{x_{i_1}, \ldots, x_{i_l}\}$ of $\{x_1, \ldots, x_k\}$, we have $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \mathcal{I}_{\{x_{i_1}, \ldots, x_{i_l}\}/\mathbb{P}^2}) = 0$.

(3.1) $(l, B) = (4, L); (8, 2L)$, where $L \in |\mathcal{O}_{\mathbb{P}^2}(1)|$.

Theorem 4. ([CF]) Let $\sigma : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at distinct 10 points $x_1, \ldots, x_{10}$ of $\mathbb{P}^2$ with the exceptional divisors $E_1, \ldots, E_{10}$. Let $L$ be a line in $\mathbb{P}^2$. Then the following are equivalent:

1. The line bundle $\hat{\mathcal{L}} = \sigma^* \mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\hat{\mathbb{P}}^2}(-E_1 - \cdots - E_{10})$ is very ample.
2. For $(l, B) = (4, L)$ and $(8, 2L)$, no distinct $l$ points of $\{x_1, \ldots, x_{10}\}$ lie on any curve linearly equivalent to $B$; and $\{x_1, \ldots, x_{10}\}$ do not lie on any member $B \in |\mathcal{O}_{\mathbb{P}^2}(3)|$.
3. For $(l, B) = (4, L); (8, 2L)$ and $(10, 3L)$, and for every distinct $l$ points $\{x_{i_1}, \ldots, x_{i_l}\}$ of $\{x_1, \ldots, x_k\}$, we have $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \mathcal{I}_{\{x_{i_1}, \ldots, x_{i_l}\}/\mathbb{P}^2}) = 0$.

References


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