VERY AMPLE LINE BUNDLES ON REGULAR SURFACES OBTAINED BY PROJECTION

Atsushi Noma*

We work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. Let $X \subseteq \mathbb{P}^N$ be a smooth irreducible surface with very ample line bundle $\mathcal{L} := \mathcal{O}_X(1)$. For points $x_1, \dots, x_m \in X$ of X, consider the blowing up $\sigma\colon \hat{X} \to X$ of X at $x_1, \dots, x_m$ with the exceptional divisors $E_1, \dots, E_m$ and the line bundle $\hat{\mathcal{L}} = \sigma^*\mathcal{L} \otimes \mathcal{O}_{\hat{X}}(-E_1 - \cdots - E_m)$ on $\hat{X}$. The problem of finding the condition for $\hat{\mathcal{L}}$ to be very ample in terms of the configuration of points $x_1, \dots, x_m \in X$ is considered by many authors ([Be],[CE],[C],[DH],[L]). This problem arises, for example, in the classification of surfaces by sectional genus (see, for example [I], [L]). In the classification of Del Pezzo surfaces, it is well-known that for $X = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(3)$, $\hat{\mathcal{L}}$ is very ample if and only if $x_1, \dots, x_m$ are distinct points with $m \le 6$ such that no three points of $x_1, \dots, x_m$ are collinear and no six points of them lie on conic in $\mathbb{P}^2$. Here along this line, we give this type of condition for surfaces with $h^1(\mathcal{O}_X) = 0$.

Theorem 1. *Let $X \subseteq \mathbb{P}^N$ $(N = d-g+1)$ be a nondegenerate, linearly normal, smooth, projective variety of dimension 2, degree d, sectional genus g, and irregularity $h^1(\mathcal{O}_X) = 0$. Let $x_1, \dots, x_m$ be distinct m points of X for $m \le d - 2g - 1$. Let $\sigma\colon \hat{X} \to X$ be a blowing up at $x_1, \dots, x_m$. Then $\hat{\mathcal{L}} = \sigma^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\hat{X}}(-E_1 - \cdots - E_m)$ is very ample if and only if for all l with $1 \le l \le m$, any l points of $\{x_1, \dots, x_m\}$ do not lie on any rational normal curve of degree l on X.*

The key ingredient of the proof is the *inner projection*, that is the linear projection from a point of X, which appears in the definition of blowing up of a point of X. By using this, for $m = 1$, $\hat{\mathcal{L}}$ is very ample if and only if there is no line through x_1 meeting X in 3 points counted with multiplicity. We apply this argument successively, based on the following lemma.

Lemma 2. *Let X be as in Theorem.*

(1) *Assume that $d = \deg X \ge 2g + 2$. If $\ell \subseteq \mathbb{P}^N$ is a line not lying on X, then $l(X \cap \ell) \le 2$.*
(2) *If $D \subseteq X$ is an irreducible reduced curve of degree m with $m \le d - 2g$, then $D \cdot E \le 1$ for a line E on X with $D \ne E$.*

As an application of Theorem, we will give a necessary and sufficient condition for line bundles to be very ample which appear in the classification by the sectional genus due to Ionescu [I, (3.1) and (4.1)]. We deal with the following three cases:

(A) $(X, \mathcal{L})$ is isomorphic to F_e, $H_e = 2C_0 + (3+e)F$, $e = 0, 1, 2$, or to the blowing-up σ of one of these with center $k \le 7$ points lying on different fibres, $H = \sigma^*(H_e) - E_1 - \cdots - E_k$; moreover $5 \le d \le 12$. $\implies$ $g = 2$: $d - 2g - 1 = 12 - 5 = 7$.
(B) $(X, \mathcal{L})$ is isomorphic to F_e, $H_e = 2C_0 + (4+e)F$, $e = 0, 1, 2, 3$, or to the blowing-up σ of one of these with center $k \le 9$ points lying on different fibres, $H = \sigma^*(H_e) - E_1 - \cdots - E_k$; moreover $7 \le d \le 16$. $\implies$ $g = 3$: $d - 2g - 1 = 16 - 7 = 9$.
(C) $\mathbb{P}^2$, $H = 4L$ where L is a line, or a blowing-up σ of it with center $k \le 10$ ordinary points, $H = \sigma^*(4L) - E_1 - \cdots - E_k$; moreover $6 \le d \le 16$. $\implies$ $g = 3$: $d - 2g - 1 = 16 - 7 = 9$.

For example, in the third case (C), we have the following:

Theorem 3. *Let $\sigma\colon \hat{\mathbb{P}^2} \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at $x_1, \dots, x_k$ with $k \leq 9$ with the exceptional divisors $E_1, \dots, E_k$. Let $\mathcal{L}$ be the line bundle $\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\hat{\mathbb{P}^2}}(-E_1 - \cdots - E_k)$. Then the following are equivalent:*

(1) *$\mathcal{L}$ is very ample.*
(2) *For every integer l and divisor B in (3.1), no distinct l points of $\{x_1, \dots, x_k\}$ lie on any curve linearly equivalent to B.*
(3) *For every integer l and divisor B in (3.1), and for every distinct l points $\{x_{i_1}, \dots, x_{i_l}\}$ of $\{x_1, \dots, x_k\}$, we have $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \mathcal{I}_{\{x_{i_1},\dots,x_{i_l}\}/\mathbb{P}^2}) = 0$.*

$$(l, B) = (4, L);\ \ (8, 2L), \qquad \text{where } L \in |\mathcal{O}_{\mathbb{P}^2}(1)|. \tag{3.1}$$

Theorem 4. ([CF]) *Let $\sigma\colon \hat{X} = \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at distinct 10 points $x_1, \dots, x_{10}$ of $\mathbb{P}^2$ with the exceptional divisors $E_1, \dots, E_{10}$. Let L be a line in $\mathbb{P}^2$. Then the following are equivalent:*

(1) *The line bundle $\hat{\mathcal{L}} = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_{\hat{\mathbb{P}^2}}(-E_1 - \cdots - E_{10})$ is very ample.*
(2) *For $(l, B) = (4, L)$ and $(8, 2L)$, no distinct l points of $\{x_1, \dots, x_{10}\}$ lie on any curve linearly equivalent to B; and $\{x_1, \dots, x_{10}\}$ do not lie on any member $B \in |\mathcal{O}_{\mathbb{P}^2}(3)|$.*
(3) *For $(l, B) = (4, L)$, $(8, 2L)$ and $(10, 3L)$, and for every distinct l points $\{x_{i_1} \dots x_{i_l}\}$ of $\{x_1, \dots, x_k\}$, we have $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \mathcal{I}_{\{x_{i_1},\dots,x_{i_l}\}/\mathbb{P}^2}) = 0$.*

References

[Be] E. Bese, *On the spannedness and very ampleness of certain line bundles on the blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ and F_r*, Math. Ann. **262** (1983), 225–238.

[CF] F.Catanese, M. Franciosi, *Divisors of small genus on algebraic surfaces and projective embeddings*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc. **9** (1996), 109–140.

[C] M.Coppens, *Very ample linear systems on blowings-up at general points of smooth projective varieties*, Pacific J. Math. **202** (2002), 313–327.

[DH] J. D'Almeida, A. Hirshowitz, *Quelques plongements projectifs nonspeciaux de surfaces rationnelles*, Math. Z. **211** (1992), 479–483.

[I] P.Ionescu, *Embedded projective varieties of small invariants*, Algebraic geometry, Bucharest 1982, Lecture Note in Math. **1056** (1984), 142–186.

[L] E.L. Livorni, *On the existence of some surfaces*, Algebraic geometry (L'Aquila, 1988), Lecture Note in Math. **1417** (1990), 155–179.

Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, Yokohama 240-8501 Japan

E-mail address: `noma@edhs.ynu.ac.jp`