UNIVERSAL ABELIAN COVERS OF CERTAIN SURFACE SINGULARITIES

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Every normal complex surface singularity with $\mathbb{Q}$-homology sphere link has a universal abelian cover. Neumann and Wahl conjectured that the universal abelian cover of a rational or minimally elliptic singularity is a complete intersection singularity defined by a system of "splice diagram equations". We will discuss the universal abelian covers and an approach to the conjecture.

Let (X, o) be a normal complex surface singularity germ and Σ its link, i.e., the boundary of a regular neighborhood of $o \in X$. We may assume that X is homeomorphic to the cone over Σ. Let Γ denote the resolution graph of (X, o). It is known that Γ and Σ determine each other ([1]). Assume that Σ is a $\mathbb{Q}$-homology sphere, or equivalently, that the exceptional set of a good resolution is a tree of rational curves. Then $G := H_1(\Sigma, \mathbb{Z})$ is finite. A morphism $(Y, o) \to (X, o)$ of germs of normal surface singularities is called a *universal abelian covering* if it induces an unramified Galois covering $Y \setminus \{o\} \to X \setminus \{o\}$ with covering transformation group G. By our assumption, the universal abelian covering $(Y, o) \to (X, o)$ uniquely exist; in fact, the link of Y is the universal abelian cover of Σ in the topological sense.

Neumann and Wahl introduced the *splice diagram equations* associated with Γ satisfying the "semigroup condition"([2], [4], [3]). Let $\widetilde{Y}$ denote the singularity defined by the splice diagram equations obtained from Γ. They proved that $\widetilde{Y}$ is an isolated complete intersection surface singularity, and that if Γ also satisfies the "congruence condition", then G acts on $\widetilde{Y}$ and the quotient $\widetilde{Y}/G$ is a normal surface singularity (it is called a *splice-quotient singularity*) with resolution graph Γ, and the quotient morphism $\widetilde{Y} \to \widetilde{Y}/G$ is the universal abelian covering. They conjectured that rational singularities and minimally elliptic singularities with $\mathbb{Q}$-homology sphere links are splice-quotient singularities. Our approach to the conjecture is as follows.

Let $\pi\colon M \to X$ be the minimal good resolution, and let $A = \bigcup_i A_i$ be the decomposition of the exceptional set $A = \pi^{-1}(o)$ into irreducible components. Let $A_{\mathbb{Z}} = \sum \mathbb{Z}A_i$ and $A_{\mathbb{Q}} = A_{\mathbb{Z}} \otimes \mathbb{Q}$. Let $\bar{A}_i \in A_{\mathbb{Q}}$ satisfy $\bar{A}_i \cdot A_j = -\delta_{ij}$. We denote by $\bar{A}_{\mathbb{Z}}$ the subgroup of $A_{\mathbb{Q}}$ generated by $\bar{A}_i$'s. Then $H_1(\Sigma, \mathbb{Z})$ is isomorphic to the group $\bar{A}_{\mathbb{Z}}/A_{\mathbb{Z}}$. We can construct an $\mathcal{O}_X$-algebra $\mathcal{A} := \bigoplus_{g \in G} \pi_* \mathcal{O}_M(D_g)$ such that $Y = \operatorname{Specan}_X \mathcal{A}$, where D_g are divisors on M, and if D_g is numerically equivalent to $C_g \in \bar{A}_{\mathbb{Z}}$, then $\{C_g \mod A_{\mathbb{Z}} | g \in G\} = \bar{A}_{\mathbb{Z}}/A_{\mathbb{Z}}$ ([6]). The algebra $\mathcal{A}$ and splice diagram equations are connected by "monomial cycles".

A component A_i is called an *end-curve* if $(A - A_i) \cdot A_i \le 1$. We denote by $\mathcal{E}(A)$ the set of end-curves. A connected component of $A - A_i$ is called a *branch* of A_i. A component A_i is called a *node* if $(A - A_i) \cdot A_i \ge 3$.

Definition 1. Let $D = \sum a_i \bar{A}_i \in \bar{A}_{\mathbb{Z}}$, $a_i \geq 0$. If $a_i = 0$ for all $A_i \notin \mathcal{E}(A)$, then we call D a *monomial cycle*. For any monomial cycle $D = \sum_{i=1}^{m} a_i \bar{A}_i$, we associate a monomial

$$x(D) := \prod_{i=1}^{m} x_i^{a_i} \in \mathbb{C}[x_1, \dots, x_m].$$

The x induces an isomorphism between the semigroup of monomial cycles and that of monomials of $x_1, \dots, x_m$.

We consider the following three conditions.

Condition 2. For any branch C of any node A_i, there exists a monomial cycle D such that $D - \bar{A}_i$ is an effective integral cycle supported on C.

Condition 3. A is star-shaped, or for any branch C of any component $A_i \notin \mathcal{E}(A)$, the fundamental cycle Z_C supported on C satisfies $Z_C \cdot A_i = 1$.

Let n_i be a positive integer such that $n_i \bar{A}_i \in A_{\mathbb{Z}}$ and suppose $C_{g_i} \equiv \bar{A}_i$ (mod $A_{\mathbb{Z}}$). Then, in $H^0(\mathcal{A})$, $H^0(\mathcal{O}_M(D_{g_i}))^{n_i} \subset H^0(\mathcal{O}_X)$.

Condition 4. For each end $A_i \in \mathcal{E}(A)$, there exists $y_i \in H^0(\mathcal{O}_M(D_{g_i}))$ such that $\operatorname{div}(y_i^{n_i})$ is of the form $n_i(\bar{A}_i + H)$, where H has no component of A and $A \cdot H = A_i \cdot H = 1$.

Condition 2 is equivalent to the semigroup condition and the congruence condition ([3]). So under this condition, splice diagram equations are defined and the "leading terms" are linear combinations of $x(D)$'s, where D's are the monomial cycles as in Condition 2.

Lemma 5. *Condition 3 implies Condition 2. Condition 3 and 4 are satisfied if (X, o) is a rational or a minimally elliptic singularity.*

Let $S = \mathbb{C}\{x_1, \dots, x_m\}$ be the convergent power series ring.

Theorem 6 ([5]). *Suppose that Condition 3 and 4 are satisfied. Then a homomorphism $\psi\colon S \to \mathcal{O}_{Y,o}$ defined by $\psi(x_i) = y_i$ is surjective, and the kernel of ψ is generated by the splice diagram equations which are homogeneous with respect to G-grading.*

Therefore, rational singularities and minimally elliptic singularities with $\mathbb{Q}$-homology sphere links are splice-quotient singularities.

References

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